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Quaternion orthogonal designs from complex companion designs

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Abstract

The success of applying generalized complex orthogonal designs as space–time block codes recently motivated the definition of quaternion orthogonal designs as potential building blocks for space–time-polarization block codes. This paper offers techniques for constructing quaternion orthogonal designs via combinations of specially chosen complex orthogonal designs. One technique is used to build quaternion orthogonal designs on complex variables for any even number of columns. A second related technique is applied to maximum rate complex orthogonal designs to generate an infinite family of quaternion orthogonal designs on complex variables such that the resulting designs have no zero entries. This second technique is also used to generate an infinite family of quaternion orthogonal designs defined over quaternion variables that display a regular redundancy. The proposed constructions are theoretically important because they provide the first known direct techniques for building infinite families of orthogonal designs over quaternion variables for any number of columns.
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1. Introduction and definitions

The development of quaternion orthogonal designs for future use in signal processing as space–time-polarization block codes has been recently introduced [1,2]. This idea was sparked by previous work in space–time coding and in polarization diversity. Space–time block codes can be viewed as a generalization of Alamouti's scheme [3] for wireless transmissions utilizing two transmit antennas. Tarokh et al. [4] generalized Alamouti's scheme by defining generalized (rectangular) complex orthogonal designs for use as complex orthogonal space–time block codes for wireless transmissions utilizing multiple transmit antennas. The resulting space–time block codes effectively combine space and time diversity and provide a major step in moving the capacity of wireless communication systems towards the theoretical limits [4,5]. The success of space–time block codes led to the idea of developing new codes that could effectively combine space and time diversity with an additional form of diversity. Polarization diversity has been shown to offer performance improvements [6–10], and the polarization states have been shown to be presentable via quaternion representations [11]. This motivated us to develop a theory of orthogonal designs over the quaternion domain for application as space–time-polarization block codes. This paper focuses on the theoretical aspects of construction methods for these newly proposed designs, while preliminary explorations of practical advantages are considered in our other work [2].

Here, we provide a brief summary of the definitions necessary to build quaternion orthogonal designs; these definitions, along with several examples, were recently introduced in our other work [2].

Definition 1. A *generalized complex orthogonal design* (*GCOD*) A of type $(s_1, s_2, \ldots, s_u)$ on commuting complex variables $z_1, \ldots, z_u$, is an $r \times n$ matrix with terms from $\{0, \pm z_1, \pm z_1^*, \ldots, \pm z_u, \pm z_u^*\}$, including possible multiplications by the complex imaginary unit $\mathbf{i}$, satisfying $A^H A = \sum_{h=1}^{u} s_h |z_h|^2 I_n$, where I_n is the $n \times n$ identity matrix, H is the Hermitian transform, and so the columns of A are formally orthogonal.

In this paper, we restrict our attention to generalized complex orthogonal designs (GCODs) where $s_h = 1$ for all h and each variable appears exactly once per column and at most once per row. In the literature, these GCODs are known as *pure*.

Definition 2. A quaternion variable $\mathbf{a} = a_1 + a_2\mathbf{i} + a_3\mathbf{j} + a_4\mathbf{k}$, where a_p, $p = 1, \ldots, 4$ are real variables and $\{\pm 1, \pm \mathbf{i}, \pm \mathbf{j}, \pm \mathbf{k}\}$ are elements of $\mathcal{Q}$, the non-commutative quaternions, has a *quaternion conjugate* defined by $\mathbf{a}^Q = a_1 - a_2\mathbf{i} - a_3\mathbf{j} - a_4\mathbf{k}$. Given a matrix $Y = (y_{\ell,m})$, where y_u are quaternion variables or numbers, its *quaternion transform* is $Y^Q = (y_{m,\ell}^Q)$.

Many properties of the quaternions are well known [12] and will be used extensively in this paper. For example, $\mathbf{a}^Q \mathbf{a} = \mathbf{a}\mathbf{a}^Q = |\mathbf{a}|^2$ is real, and any quaternion variable $\mathbf{a}$ can be written as $\mathbf{a} = z_1 + z_2\mathbf{j}$, where z_1, z_2 are appropriately chosen complex variables.

We now review two definitions for quaternion orthogonal designs recently introduced by some of the current authors [2]. These definitions parallel the definitions of real and complex orthogonal designs and their generalizations [4,5,13,14].

Definition 3. A *quaternion orthogonal design (QOD) of type* $(s_1, s_2, \ldots, s_u)$ *on complex variables* $z_1, z_2, \ldots, z_u$ is an $r \times n$ matrix A with terms from $\{0, \pm z_1, \pm z_1^*, \pm z_2, \pm z_2^*, \ldots, \pm z_u, \pm z_u^*\}$ including possible multiplications on the left and/or right by quaternion elements $\mathbf{q} \in \mathcal{Q} = \{\pm 1, \pm \mathbf{i}, \pm \mathbf{j}, \pm \mathbf{k}\}$ such that $A^Q A = \left(\sum_{h=1}^{u} s_h |z_h|^2\right) I_n$, so the columns of A are formally orthogonal.

Example 1. Consider the QOD $A = \begin{bmatrix} \mathbf{i}z_1 & \mathbf{i}z_2 \\ -\mathbf{j}z_2^* & \mathbf{j}z_1^* \end{bmatrix}$, where z_1, z_2 are commuting complex variables. Expanding the complex variables in A using $z_h = x_h + y_h \mathbf{i}$, where the x_h, y_h are real variables gives $A = \begin{bmatrix} -y_1 + \mathbf{i}x_1 & -y_2 + \mathbf{i}x_2 \\ -\mathbf{j}x_2 - \mathbf{k}y_2 & \mathbf{j}x_1 + \mathbf{k}y_1 \end{bmatrix}$. We can view the entries of A as quaternion variables such that certain components of the variables are restricted to zero. For example, the (2,2) position represents a quaternion variable $\mathbf{a} = a_1 + a_2 \mathbf{i} + a_3 \mathbf{j} + a_4 \mathbf{k}$ such that $a_1 = a_2 = 0$, $a_3 = x_1$, and $a_4 = y_1$ [2].

Definition 4. A *quaternion orthogonal design (QOD) of type* $(s_1, s_2, \ldots, s_u)$ *on quaternion variables* $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_u$ is an $r \times n$ matrix A with terms from $\{0, \pm \mathbf{a}_1, \pm \mathbf{a}_1^Q, \pm \mathbf{a}_2, \pm \mathbf{a}_2^Q, \ldots, \pm \mathbf{a}_u, \pm \mathbf{a}_u^Q\}$ including possible multiplications on the left and/or right by quaternion elements $\mathbf{q} \in \mathcal{Q}$ and to satisfy $A^Q A = \left(\sum_{h=1}^{u} s_h |\mathbf{a}_h|^2\right) I_n$, so the columns of A are formally orthogonal.

It is clear that QODs on complex variables are a restricted form of QODs on quaternion variables. We can generalize these designs by allowing the design entries to be real linear combinations of the permitted variables and their quaternion multipliers, in which case we say the design is *with linear processing*. We say that an orthogonal design is *zero-free* if it contains no entries that are zero.

2. Companion and zero-masking companion matrices

In this section, we introduce column companion matrices, row companion matrices, and zero-masking companion matrices. We will use these constructs to build QODs in Sections 3 and 4.

Definition 5. Let A be an $r \times n$ generalized complex orthogonal design. We define a *column (row) companion matrix* B by permuting the columns (rows) of A according to a permutation π with order 2, which implies that $\pi^2 = 1$ and $\pi^{-1} = \pi$.

Clearly, for a column (row) companion matrix to exist for an $r \times n$ GCOD, n (r) must be even. Furthermore, as shown by Liang [5] and Lu et al. [17], a GCOD of size $r \times n$ exists for all n, and in particular for all even n. Hence, for any even n, we may build a GCOD A of size $r \times n$ and an associated column companion matrix B. For example, consider the permutation π with cycle notation $\pi = (12)(34) \cdots (n-1, n)$. This permutation has order 2 so that $\pi^{-1} = \pi$. Then, if $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n$ denote the n columns of A in order, then the columns of B are $\mathbf{c}_2, \mathbf{c}_1, \mathbf{c}_4, \mathbf{c}_3, \ldots, \mathbf{c}_n, \mathbf{c}_{n-1}$ in order.

Example 2. The following generalized complex orthogonal design A achieves the maximum rate of 3/4 [5], but has twice the achievable minimum decoding delay [15]. It is formed using Liang's

algorithm [5], then rearranging the order in which the rows appear and multiplying certain rows by -1.

$$A = \begin{bmatrix} z_1 & 0 & z_2 & z_3 \\ 0 & z_1 & z_4 & z_5 \\ -z_2^* & -z_4^* & z_1^* & 0 \\ -z_3^* & -z_5^* & 0 & z_1^* \\ \hline z_6^* & 0 & -z_5^* & z_4^* \\ 0 & z_6^* & z_3^* & -z_2^* \\ z_5 & -z_3 & z_6 & 0 \\ -z_4 & z_2 & 0 & z_6 \end{bmatrix}.$$

One column companion matrix for A, formed by applying the permutation $\pi = (13)(24)$ to the columns of A, is as follows:

$$B_{\text{Col}} = \begin{bmatrix} z_2 & z_3 & z_1 & 0 \\ z_4 & z_5 & 0 & z_1 \\ z_1^* & 0 & -z_2^* & -z_4^* \\ 0 & z_1^* & -z_3^* & -z_5^* \\ \hline -z_5^* & z_4^* & z_6^* & 0 \\ z_3^* & -z_2^* & 0 & z_6^* \\ z_6 & 0 & z_5 & -z_3 \\ 0 & z_6 & -z_4 & z_2 \end{bmatrix}.$$

One row companion matrix for A, formed by applying the permutation $\pi = (12)(34)(56)(78)$ to the rows of A, is as follows:

$$B_{\text{Row}} = \begin{bmatrix} 0 & z_1 & z_4 & z_5 \\ z_1 & 0 & z_2 & z_3 \\ -z_3^* & -z_5^* & 0 & z_1^* \\ -z_2^* & -z_4^* & z_1^* & 0 \\ \hline 0 & z_6^* & z_3^* & -z_2^* \\ z_6^* & 0 & -z_5^* & z_4^* \\ -z_4 & z_2 & 0 & z_6 \\ z_5 & -z_3 & z_6 & 0 \end{bmatrix}.$$

Definition 6. Let A be an $r \times n$ generalized complex orthogonal design. We say that A is *zero-maskable by columns (rows)* if the columns (rows) of A can be paired together in mutually exclusive pairs such that the two columns (rows) in a given pair are never both zero in the same row (column). If we form a column (row) companion matrix B whose columns (rows) are the zero-masking partners of the respective columns (rows) of A, then B is called a *zero-masking column (row) companion* for A.

Example 3. Referring to the designs in Example 2, the design B_{Col} is a zero-masking column companion for A, and the design B_{Row} is a zero-masking row companion for A.

3. The column companion $A + B\mathbf{j}$ construction

In this section, we use column companion matrices and zero-masking column companion matrices to build QODs on complex variables.

Theorem 1. *The existence of an $r \times 2n$ generalized complex orthogonal design on complex variables $z_1, z_2, \ldots, z_u$ implies the existence of an $r \times 2n$ quaternion orthogonal design with linear processing on complex variables $z_1, z_2, \ldots, z_u$.*

Due to the construction algorithms provided by Liang [5] and Lu et al. [17], Theorem 1 implies that there exist quaternion orthogonal designs with linear processing on complex variables for any even number of columns.

Proof. We write $A = [A_1 \quad A_2]$ for the $r \times 2n$ GCOD on the u commuting complex variables, $z_1, \ldots, z_u$, where A_1 and A_2 are both of size $r \times n$ on the u commuting complex variables. Then, by definition, $A^H A = \left(\sum_{h=1}^{u} |z_h|^2\right) I_{2n}$. Thus

$$A^H A = \begin{bmatrix} A_1^H \\ A_2^H \end{bmatrix} \begin{bmatrix} A_1 & A_2 \end{bmatrix}$$

$$= \begin{bmatrix} A_1^H A_1 & A_1^H A_2 \\ A_2^H A_1 & A_2^H A_2 \end{bmatrix}$$

$$= \left(\sum_{h=1}^{u} |z_h|^2\right) I_{2n}.$$

Hence

$$A_1^H A_2 = 0 = A_2^H A_1$$

and

$$A_1^H A_1 = A_2^H A_2 = \left(\sum_{h=1}^{u} |z_h|^2\right) I_n.$$

We note $\left(\sum_{h=1}^{u} |z_h|^2\right) I_n$ is real.

Now, since permuting the columns of $A = [A_1 \quad A_2]$ preserves the orthogonality of the matrix, we construct another GCOD $B = [A_2 \quad A_1]$. Note that B is a column companion matrix for A formed by applying the permutation $(1, n+1)(2, n+2) \cdots (n, 2n)$ to the columns of A.

Then

$$B^H B = \begin{bmatrix} A_2^H \\ A_1^H \end{bmatrix} \begin{bmatrix} A_2 & A_1 \end{bmatrix}$$

$$= \begin{bmatrix} A_2^H A_2 & A_2^H A_1 \\ A_1^H A_2 & A_1^H A_1 \end{bmatrix}$$

$$= \left(\sum_{h=1}^{u} |z_h|^2\right) I_{2n}.$$

Also

$$
B^H A = \begin{bmatrix} A_2^H \\ A_1^H \end{bmatrix} \begin{bmatrix} A_1 & A_2 \end{bmatrix}
$$

$$
= \begin{bmatrix} A_2^H A_1 & A_2^H A_2 \\ A_1^H A_1 & A_1^H A_2 \end{bmatrix}
$$

$$
= \begin{bmatrix} 0 & \left(\sum_{h=1}^{u} |z_h|^2 \right) I_n \\ \left(\sum_{h=1}^{u} |z_h|^2 \right) I_n & 0 \end{bmatrix}
$$

$$
= \begin{bmatrix} A_1^H A_2 & A_1^H A_1 \\ A_2^H A_2 & A_2^H A_1 \end{bmatrix}
$$

$$
= A^H B,
$$

which shows that $B^H A$ and $A^H B$ are real and equal.

We now claim that $D = \frac{1}{\sqrt{2}}(A + B\mathbf{j})$ is the required QOD. Notice that

$$
D^Q D = \frac{1}{2}(A^Q - \mathbf{j}B^Q)(A + B\mathbf{j})
$$

$$
= \frac{1}{2}(A^Q A - \mathbf{j}B^Q B\mathbf{j} - \mathbf{j}B^Q A + A^Q B\mathbf{j}).
$$

However, as A and B are over the complex domain, we have $A^Q = A^H$ and $B^Q = B^H$. Thus $A^Q A = A^H A = \sum_{h=1}^{u} |z_h|^2 I_n$ which is real. Similarly $B^Q B$ is real and so $-\mathbf{j}B^Q B\mathbf{j} = -\mathbf{j}B^H B\mathbf{j} = -\mathbf{j} \left(\sum_{h=1}^{u} |z_h|^2 \right) I_n \mathbf{j} = \sum_{h=1}^{u} |z_h|^2 I_n$. We also have $-\mathbf{j}B^Q A + A^Q B\mathbf{j} = -\mathbf{j}B^H A + A^H B\mathbf{j} = 0$ as both $A^H B$ and $B^H A$ are shown above to be real and equal.

So we have

$$
D^Q D = \frac{1}{2} \left(\sum_{h=1}^{u} |z_h|^2 I_n + \sum_{h=1}^{u} |z_h|^2 I_n \right)
$$

$$
= \sum_{h=1}^{u} |z_h|^2 I_n
$$

So D satisfies the defining orthogonality constraint required for QODs. Furthermore, the entries of D are of the form $\frac{1}{\sqrt{2}}(z_h)$, $\frac{1}{\sqrt{2}}(z_h + \mathbf{j}z_m)$, or $\frac{1}{\sqrt{2}}(\mathbf{j}z_m)$, where the z_h and z_m are complex variables, up to conjugation and sign. So, each entry is a real linear combination of terms of the form $\mathbf{q}z$, where $\mathbf{q} \in \mathcal{Q}$ and z is a complex variable. So, D satisfies all conditions of an QOD with linear processing over complex variables $z_1, z_2, \ldots, z_u$.

Thus we have constructed our required QOD with linear processing over complex variables. $\qquad\square$

Corollary 1. *If A and B are generalized complex orthogonal designs on complex variables $z_1, z_2, \ldots, z_u$ such that $A^H B$ and $B^H A$ are real and equal, then $A + B\mathbf{j}$ is a quaternion orthogonal design on complex variables $z_1, z_2, \ldots, z_u$.*

Corollary 1 suggests a line of research in identifying pairs of GCODs A and B such that $A^H B = B^H A$ is real. This condition is familiar from the definition of *complex amicable* designs. Amicability of orthogonal designs was first considered in the real case by current author Seberry in the 1970s [18,19]. Many questions regarding complex amicability remain unsolved, and we hope that the current context will motivate more work in this area.

Example 4. We now apply the construction technique described in Theorem 1 to the GCOD A and its zero-masking column companion B_{Col} from Example 2. To emphasize the structure and allow for generalizations, we set

$$X = \begin{bmatrix} z_2 & z_3 \\ z_4 & z_5 \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} -z_5^* & z_4^* \\ z_3^* & -z_2^* \end{bmatrix},$$

so that, with I_2 the 2×2 identity matrix, we can rewrite A and B_{Col} as

$$A = \begin{bmatrix} z_1 I_2 & X \\ -X^H & z_1^* I_2 \\ z_6^* I_2 & Y \\ -Y^H & z_6 I_2 \end{bmatrix} \quad \text{and} \quad B_{\text{Col}} = \begin{bmatrix} X & z_1 I_2 \\ z_1^* I_2 & -X^H \\ Y & z_6^* I_2 \\ z_6 I_2 & -Y^H \end{bmatrix}.$$

Since $A^H A = \sum_{h=1}^{6} |z_h|^2 I_4$, we have $X^H X + Y^H Y = X X^H + Y Y^H = \sum_{h=1}^{6} |z_h|^2 I_2$. Now performing the construction technique, we get

$$D = \frac{1}{\sqrt{2}}(A + B_{\text{Col}}\mathbf{j})$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} z_1 I_2 + X\mathbf{j} & X + z_1\mathbf{j}I_2 \\ -X^H + z_1^*\mathbf{j}I_2 & z_1^* I_2 - X^H\mathbf{j} \\ z_6^* I_2 + Y\mathbf{j} & Y + z_6^*\mathbf{j}I_2 \\ -Y^H + z_6\mathbf{j}I_2 & z_6 I_2 - Y^H\mathbf{j} \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \left[\begin{array}{cccc} z_1 + z_2\mathbf{j} & z_3\mathbf{j} & z_2 + z_1\mathbf{j} & z_3 \\ z_4\mathbf{j} & z_1 + z_5\mathbf{j} & z_4 & z_5 + z_1\mathbf{j} \\ -z_2^* + z_1^*\mathbf{j} & -z_4^* & z_1^* - z_2^*\mathbf{j} & -z_4^*\mathbf{j} \\ -z_3^* & -z_5^* + z_1^*\mathbf{j} & -z_3^*\mathbf{j} & z_1^* - z_5^*\mathbf{j} \\ \hline z_6^* - z_5^*\mathbf{j} & z_4^*\mathbf{j} & -z_5^* + z_6^*\mathbf{j} & z_4^* \\ z_3^*\mathbf{j} & z_6^* - z_2^*\mathbf{j} & z_3^* & -z_2^* + z_6^*\mathbf{j} \\ z_5 + z_6\mathbf{j} & -z_3 & z_6 + z_5\mathbf{j} & -z_3\mathbf{j} \\ -z_4 & z_2 + z_6\mathbf{j} & -z_4\mathbf{j} & z_6 + z_2\mathbf{j} \end{array} \right].$$

This is a QOD with linear processing on the complex variables $z_1, z_2, \ldots, z_6$, as the entries are real linear combinations of these complex variables and their quaternion multipliers. This example shows that since there exists a zero-maskable by columns 8×4 generalized complex orthogonal design, there also exists a zero-free 8×4 quaternion orthogonal design on complex variables.

We note that the construction in Theorem 1 can also be applied to the maximum rate and minimum decoding delay design $A' = \begin{bmatrix} z_1 & 0 & z_2 & z_3 \\ 0 & z_1 & z_3^* & -z_2^* \\ -z_2^* & -z_3 & z_1^* & 0 \\ -z_3^* & z_2 & 0 & z_1^* \end{bmatrix}$ to produce a zero-free 4×4 quaternion orthogonal design on complex variables. However, we utilized the 8×4 example A obtained via the well-known Liang algorithm [5] to show that minimum decoding delay is not necessary in this construction and for further use in Section 4.3.

4. The row companion $A + \mathbf{j}B$ construction

In Section 4.1, we present a quaternion construction technique using row companion matrices and zero-masking row companion matrices. In Section 4.2, we apply this construction technique to produce zero-free QODs on complex variables, while in Section 4.3, we apply this construction technique to produce QODs on quaternion variables.

4.1. The construction

Theorem 2. *Let A be an $r \times n$ generalized complex orthogonal design, where r is even. Form a row companion matrix B by applying a permutation π of order 2 to the rows of A. Then $A + \mathbf{j}B$ is an $r \times n$ quaternion orthogonal design.*

Before providing the proof, we note for clarity that the construction in Section 3 utilizes $A + B\mathbf{j}$, while the construction in this section utilizes $A + \mathbf{j}B$. Since the quaternion $\mathbf{j}$ does not commute with the complex entries in B, it is important to make this distinction. Also, in this section and thereafter, we dropped the scaling constant of $\frac{1}{\sqrt{2}}$, since it is not essential to the discussion.

Proof. Suppose that A is an $r \times n$ generalized complex orthogonal design with r even, and suppose that B is a row companion matrix for A. We will see that the main task in this proof is to show that $A^\mathrm{T}B = B^\mathrm{T}A$, where T denotes the matrix transpose. In other words, we must show that $A^\mathrm{T}B$ is a symmetric matrix: $A^\mathrm{T}B = (A^\mathrm{T}B)^\mathrm{T} = B^\mathrm{T}A$.

We begin by proving the theorem for a special case. Suppose that B has been formed by applying the order 2 permutation $(12)(34)\cdots(r-1, r)$ to the rows of A.

Let $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n$ denote the n columns of A, in order. When expanded to show components, write $\mathbf{c}_\ell = (c_{\ell,1}, c_{\ell,2}, \ldots, c_{\ell,r})$. Since π acts on all rows of A by reordering the rows, it is clear that the effect of π on any certain column of A is a reordering of the elements in that column. We use the notation $\pi(\mathbf{c}_\ell)$ to indicate the effect that π has on a column $\mathbf{c}_\ell$ of A. So, we can now write the ℓth column of B as $\pi(\mathbf{c}_\ell)$, where $\mathbf{c}_\ell$ is the ℓth column of A.

Now, consider the (ℓ, m) entry of the product $A^\mathrm{T}B$. It is computed by taking the dot product of the ℓth column of A with the mth column of B. We can represent this as $\mathbf{c}_\ell \cdot \pi(\mathbf{c}_m)$. Similarly, consider the (m, ℓ) entry of the product $A^\mathrm{T}B$. It is computed by taking the dot product of the mth column of A with the ℓth column of B. We can represent this as $\mathbf{c}_m \cdot \pi(\mathbf{c}_\ell)$.

In order to show that $A^\mathrm{T}B$ is symmetric, we must show that $\mathbf{c}_\ell \cdot \pi(\mathbf{c}_m) = \mathbf{c}_m \cdot \pi(\mathbf{c}_\ell)$. Notice that

$$\mathbf{c}_\ell \cdot \pi(\mathbf{c}_m) = \pi(\mathbf{c}_m) \cdot \mathbf{c}_\ell \text{ (dot product of complex variables is commutative)}$$

$$= (c_{m,2}, c_{m,1}, c_{m,4}, c_{m,3}, \ldots, c_{m,r}, c_{m,r-1}) \cdot (c_{\ell,1}, c_{\ell,2}, \ldots, c_{\ell,r})$$

$$\text{(using definition of } \pi)$$

$$= c_{m,2}c_{\ell,1} + c_{m,1}c_{\ell,2} + c_{m,4}c_{\ell,3} + c_{m,3}c_{\ell,4} + \cdots + c_{m,r}c_{\ell,r-1} + c_{m,r-1}c_{\ell,r}$$

$$= c_{m,1}c_{\ell,2} + c_{m,2}c_{\ell,1} + c_{m,3}c_{\ell,4} + c_{m,4}c_{\ell,3} + \cdots + c_{m,r-1}c_{\ell,r}$$

$$+ c_{m,r}c_{\ell,r-1} \text{ (reordering of terms)}$$

$$= \mathbf{c}_m \cdot \pi(\mathbf{c}_\ell).$$

So, the (ℓ, m) entry of $A^\mathrm{T}B$ is equal to the (m, ℓ) entry of $A^\mathrm{T}B$. Hence, we have shown that $A^\mathrm{T}B$ is symmetric and so $A^\mathrm{T}B = B^\mathrm{T}A$. With this relationship, we now have:

$$
\begin{aligned}
(A + \mathbf{j}B)^Q (A + \mathbf{j}B) &= (A^Q + (\mathbf{j}B)^Q)(A + \mathbf{j}B) \\
&= (A^Q - B^Q\mathbf{j})(A + \mathbf{j}B) \\
&= A^Q A - (B^Q\mathbf{j})(\mathbf{j}B) + A^Q(\mathbf{j}B) - (B^Q\mathbf{j})A \\
&= A^Q A + B^Q B + (\mathbf{j}A^{\mathrm{T}})(B) - (\mathbf{j}B^{\mathrm{T}})(A) \\
&= A^Q A + B^Q B + \mathbf{j}(A^{\mathrm{T}}B - B^{\mathrm{T}}A) \\
&= A^H A + B^H B + \mathbf{j}(A^{\mathrm{T}}B - B^{\mathrm{T}}A)
\end{aligned}
\tag{1}
$$

(since A and B are complex)

$$
= \sum_{h=1}^{u} z_h z_h^* I + \sum_{h=1}^{u} z_h z_h^* I
$$

(since A and B are orthogonal designs and since $A^{\mathrm{T}}B = B^{\mathrm{T}}A$)

$$
= \sum_{h=1}^{u} 2|z_h|^2 I.
$$

To see (1), note for example that $A^Q(\mathbf{j}B) = (A^Q\mathbf{j})B$, and then represent the (ℓ, m) entry of matrix A^Q as z^*, where z is a complex variable in entry (m, ℓ) in A. Then, since $z^*\mathbf{j} = \mathbf{j}z$, we have the (ℓ, m) entry of $A^Q\mathbf{j}$ equal to $\mathbf{j}z$, equivalently $\mathbf{j}$ times the (m, ℓ) entry of A. Hence, $A^Q\mathbf{j} = \mathbf{j}A^{\mathrm{T}}$.

So, we have shown that when the row companion matrix B is formed from A using $\pi = (12)(34)\cdots(r-1, r)$, $A^{\mathrm{T}}B$ is symmetric and thus $A + \mathbf{j}B$ is a quaternion orthogonal design.

We now show that the result follows for any choice of order 2 permutation, π. To see this, note that in any case, the mth column of B is $\pi(\mathbf{c}_m)$, where π is *some* permutation with order 2. Then, the mth column of B, $\pi(\mathbf{c}_m)$ is expanded as $(c_{m,\pi(1)}, c_{m,\pi(2)}, \ldots, c_{m,\pi(r)})$, where $c_{m,\pi(h)}$ represents the image of the hth component of $\mathbf{c}_m$ under the permutation by π.

We have

$$
\begin{aligned}
\mathbf{c}_\ell \cdot \pi(\mathbf{c}_m) &= \pi(\mathbf{c}_m) \cdot \mathbf{c}_\ell \text{ (dot product of complex variables is commutative)} \\
&= (c_{m,\pi(1)}, c_{m,\pi(2)}, \ldots, c_{m,\pi(r)}) \cdot (c_{\ell,1}, c_{\ell,2}, \ldots, c_{\ell,r}) \\
&= \pi(c_{m,\pi(1)}, c_{m,\pi(2)}, \ldots, c_{m,\pi(r)}) \cdot \pi(c_{\ell,1}, c_{\ell,2}, \ldots, c_{\ell,r}) \\
&= \mathbf{c}_m \cdot \pi(\mathbf{c}_\ell) \text{ (since } \pi^2 = 1).
\end{aligned}
\tag{2}
$$

To see (2), notice that for any two vectors $\mathbf{x}, \mathbf{y}$ of complex variables, the dot product $\mathbf{x} \cdot \mathbf{y}$ is the sum of the products of the corresponding components of $\mathbf{x}$ and $\mathbf{y}$. So, summing these products in any order produces the same result. Applying the same permutation π to each vector simply permutes the order in which the intermediary products are summed. So, $\mathbf{x} \cdot \mathbf{y}$ is equal to $\pi(\mathbf{x}) \cdot \pi(\mathbf{y})$, for any permutation π.

We have now shown given any generalized complex orthogonal design A with an even number of rows and a row companion matrix B, $A^{\mathrm{T}}B$ is symmetric. So, in all cases, $A + \mathbf{j}B$ is an orthogonal design over the quaternion domain.   □

The proof of Theorem 2 gives the following useful corollary:

Corollary 2. *If A and B are generalized complex orthogonal designs such that $A^{\mathrm{T}}B$ is symmetric, then $A + \mathbf{j}B$ is a quaternion orthogonal design.*

Corollary 2 suggests future work in determining the exact conditions on complex orthogonal designs A and B such that $A^{\mathrm{T}}B$ is symmetric. We note again the similarity of this condition with the definition of amicability.

4.2. Zero-free quaternion orthogonal designs

In this section, we use Theorem 2 to build zero-free QODs with linear processing on complex variables. Specifically, given a maximum rate GCOD A with an even number of columns $2n$, we show that it is possible to form a row companion matrix B so that $A + \mathbf{j}B$ is a *zero-free* QOD with $2n$ columns. Since the deletion of any column within a QOD is also a QOD, this construction produces zero-free QODs for any number of columns.

Corollary 3. *Let A be an $r \times n$ generalized complex orthogonal design of type $(1, 1, \ldots, 1)$, where r is even. If A is zero maskable by rows, then let B be a zero-masking row companion for A. Then $A + \mathbf{j}B$ is a zero-free quaternion orthogonal design of type $(2, 2, \ldots, 2)$ with linear processing.*

Example 5. Using A and the zero-masking row companion B_{Row} from Example 2, we obtain the following zero-free design:

$$A + \mathbf{j}B_{\mathrm{Row}} = \left[\begin{array}{cccc|} z_1 & \mathbf{j}z_1 & z_2 + \mathbf{j}z_4 & z_3 + \mathbf{j}z_5 \\ \mathbf{j}z_1 & z_1 & z_4 + \mathbf{j}z_2 & z_5 + \mathbf{j}z_3 \\ -z_2^* - \mathbf{j}z_3^* & -z_4^* - \mathbf{j}z_5^* & z_1^* & \mathbf{j}z_1^* \\ -z_3^* - \mathbf{j}z_2^* & -z_5^* - \mathbf{j}z_4^* & \mathbf{j}z_1^* & z_1^* \\ \hline z_6^* & \mathbf{j}z_6^* & -z_5^* + \mathbf{j}z_3^* & z_4^* - \mathbf{j}z_2^* \\ \mathbf{j}z_6^* & z_6^* & z_3^* - \mathbf{j}z_5^* & -z_2^* + \mathbf{j}z_4^* \\ z_5 - \mathbf{j}z_4 & -z_3 + \mathbf{j}z_2 & z_6 & \mathbf{j}z_6 \\ -z_4 - \mathbf{j}z_5 & z_2 - \mathbf{j}z_3 & \mathbf{j}z_6 & z_6 \end{array} \right].$$

This resulting QOD $A + \mathbf{j}B_{\mathrm{Row}}$ is with linear processing, as the entries are permitted to be linear combinations of terms of the form $\mathbf{q}z$, where $\mathbf{q} \in \mathscr{Q}$ and z is a complex variable.

We will now apply Corollary 3 to maximum rate GCODs. Recall that the rate of a GCOD is defined to be the ratio of the number of complex variables to the number of rows. In 2003, Liang proved that the maximum rate of a GCOD with $2n - 1$ or $2n$ columns is $\frac{n+1}{2n}$ [5]. Recall that the minimum decoding delay for a given number of columns is defined as the minimum number of rows necessary to achieve a maximum rate design. Recently, some of the current authors proved that a tight lower bound on the minimum decoding delay for a maximum rate GCOD with $2n - 1$ or $2n$ columns is $\widetilde{r_n} = \binom{2n}{n-1}$ [15]. When the number of columns $2n - 1$ or $2n$ is congruent to 0, 1, or 3 modulo 4, the maximum rate GCOD can achieve the lower bound of $\binom{2n}{n-1}$ [17]. When the number of columns is congruent to 2 modulo 4, the GCOD can achieve $2\binom{2n}{n-1}$, twice the lower bound [5,17].

We are now ready to provide our next result:

Theorem 3. *Let A be a $\kappa\binom{2n}{n-1} \times 2n$ maximum rate generalized complex orthogonal design, where $\kappa \in \{1, 2\}$ is chosen so that $\kappa\binom{2n}{n-1}$ is even. Then it is possible to form a zero-masking*

row companion matrix B, so that $A + \mathbf{j}B$ is a zero-free quaternion orthogonal design with linear processing.

Proof. For each n, $\kappa \binom{2n}{n-1}$ can be shown through algebraic manipulations to be even for at least one value of $\kappa \in \{1, 2\}$. Let A be a maximum rate GCOD with $2n$ columns and $\kappa \binom{2n}{n-1}$ rows, where $\kappa \in \{1, 2\}$ is chosen so that $\kappa \binom{2n}{n-1}$ is even. Such designs can be constructed for all n using well-known algorithms [5,17]. In the cases where such designs exist for both values of κ, $\kappa = 1$ is preferred for applications as these designs achieve the minimum decoding delay.

Theorem 2 then shows that given any row companion matrix B for A, the matrix $A + \mathbf{j}B$ is a QOD. Corollary 3 shows that given any zero-masking row companion matrix B for A, the matrix $A + \mathbf{j}B$ is a zero-free QOD. So, it remains to show that we can always construct a zero-masking companion for any maximum rate GCOD with $2n$ columns and $\kappa \binom{2n}{n-1}$ rows, where $\kappa \in \{1, 2\}$ is chosen so that $\kappa \binom{2n}{n-1}$ is even. In other words, we must show that the rows of A can be partitioned into mutually exclusive pairs so that no two rows in any given pair are zero in the same column. This can be directly confirmed for the small cases with 2 and 4 columns. Therefore, we examine the case of greater than 4 columns.

Liang showed that for any variable z_i in a GCOD A, it is possible to rearrange the matrix through suitable row and column rearrangements and multiplications of rows or columns by -1 to obtain a submatrix:

$$B_\ell = \begin{pmatrix} z_\ell & 0 & \cdots & 0 & & & \\ 0 & z_\ell & \cdots & 0 & & M_\ell & \\ \vdots & \vdots & \ddots & \vdots & & & \\ 0 & 0 & \cdots & z_\ell & & & \\ \hline & & & & z_\ell^* & 0 & \cdots & 0 \\ & & & & 0 & z_\ell^* & \cdots & 0 \\ & -M_\ell^H & & & \vdots & \vdots & \ddots & \vdots \\ & & & & 0 & 0 & \cdots & z_\ell^* \end{pmatrix},$$

where B_ℓ is $2n \times 2n$ when A has $2n$ columns [5]. Liang showed that for maximum rate GCODs with $2n$ columns, M_ℓ is $n \times n$, has no zero entries, and for $n > 2$, the variables in M_ℓ are distinct (without considering conjugation or sign as distinctions) [5].

We say that a maximum rate GCOD A with $2n$ columns is in "B_ℓ form" if the submatrix B_ℓ (with its $n \times n$ submatrix M_ℓ) can be created in A through only rearranging the order in which the rows appear ("row swaps"), possible multiplications by -1, and possible conjugations of all instances of z_ℓ. By prior results, we may assume that any rearrangement of the columns of A is in B_ℓ form for some ℓ [15]. We let X_ℓ denote the first n rows of B_ℓ and let Y_ℓ denote the last n rows of B_ℓ.

By the structure of the submatrix B_ℓ, clearly any row within X_ℓ is zero masked by any row within Y_ℓ. In any manner, form mutually exclusive pairs between rows in X_ℓ and rows in Y_ℓ. As the number of rows in B_ℓ is even, namely $2n$, and the number of rows in X_ℓ is equal to the number of rows in Y_ℓ, any such pairing will always work.

Now, keeping these pairings on record, swap any column from the first n columns of A with any of the last n columns of A. Such a column swap affects all rows in A by the same permutation

of coordinates. We have previously shown that such a swap moves A into B_m form, and we may assume (through appropriate such column swaps) that $m \neq \ell$ [15]. For convenience of discussion, perform any necessary row swaps and multiplications by -1 until B_m appears at the top of A. Now, consider the rows in B_m that did not appear (up to permutation) in B_ℓ. These rows are currently unpaired, as they were not considered in the first iteration of row pairing. We must show that the number of unpaired rows in X_m is exactly equal to the number of unpaired rows in Y_m, since this would allow us to form zero-masked pairs between rows in X_m and rows in Y_m.

We must consider the situation where a row (up to permutation) is contained in both B_ℓ and B_m. If certain permutations of a given row $\mathbf{r}$ appear in both B_ℓ and B_m, $\ell \neq m$, we say that B_ℓ and B_m "share" the row $\mathbf{r}$. We now show that if B_ℓ and B_m share a row $\mathbf{r}$, then they must share exactly two rows: Suppose $\mathbf{r}$ is shared by B_ℓ and B_m. For convenience, we follow Liang's convention to say that an entry of $\pm z_\ell$ or $\pm z_\ell^*$ has color ℓ [5]. Then, since each row in B_ℓ contains an element of color ℓ and since each row in B_m contains an element of color m, this row $\mathbf{r}$ contains one variable of color ℓ and one of color m. (Recall that no variable of any color appears more than once in any row, by the definition of a GCOD.)

Now, consider the permutation of $\mathbf{r}$ that appears in B_ℓ. Due to the structure of B_ℓ, the element of color ℓ must appear along the main diagonal of B_ℓ and the variable of color m must appear within M_ℓ or within $-M_\ell^H$. Suppose without loss of generality that it appears within M_ℓ. Then, this row $\mathbf{r}$ must appear within X_ℓ, the top half of B_ℓ. Then, since B_ℓ contains both submatrices M_ℓ and $-M_\ell^H$, there must be a second row $\mathbf{r}'$ in B_ℓ that contains another variable of color m in $-M_\ell^H$. Clearly, this second element of color m will have the opposite sign and conjugation of the originally considered element of color m, and this second row $\mathbf{r}'$ must be within Y_ℓ. Since every row in B_ℓ contains an element of color ℓ, this row $\mathbf{r}'$ must also contain an element of color ℓ. We now claim that, under the same permutation that puts $\mathbf{r}$ in B_m, $\mathbf{r}'$ is also put in B_m: This is true since it has been shown that every instance of z_m (up to conjugation and sign) must appear in B_m [15]. For the same reasoning that placed (say) $\mathbf{r}$ in X_ℓ and $\mathbf{r}'$ in Y_ℓ, we know that within B_m the permutation of one of these rows appears in X_m and the permutation of the other appears in Y_m. Specifically, we know that (up to signs) one of these rows contains z_m (and thus appears in X_m) and the other contains z_m^* (and thus appears in Y_m).

So, we have shown that if B_ℓ and B_m share one row, then they share (at least) two rows. We claim that B_ℓ and B_m cannot share more than two rows: For more than four columns, if B_ℓ and B_m shared more than two rows, then this would require, for example, more than one variable of color m inside M_ℓ. This would contradict Liang's result that the variables in M_ℓ have distinct colors for any maximum rate design with more than four columns [5].

So, we can conclude that if B_ℓ and B_m, $\ell \neq m$, share one row, then they share exactly two rows. Furthermore, we showed that one of the shared rows appears in X_ℓ (or X_m) and the other appears in Y_ℓ (or Y_m).

Now, suppose $\mathbf{r}$ is a row in B_ℓ, and without loss of generality assume it is in X_ℓ. While pairing together the rows in B_ℓ, $\mathbf{r}$ is paired together with some row in Y_ℓ. Then, suppose a permutation of $\mathbf{r}$ appears again in B_m. Our arguments above show that there is also some second row $\mathbf{r}'$ that is shared by B_ℓ and B_m. Furthermore, as shown above, since $\mathbf{r}$ appears in X_ℓ, $\mathbf{r}'$ must appear in Y_ℓ. So, during the handling of B_ℓ, $\mathbf{r}$ in X_ℓ was paired with some row in Y_ℓ and $\mathbf{r}'$ in Y_ℓ was paired with some row in X_ℓ. (It is even possible that $\mathbf{r}$ and $\mathbf{r}'$ were paired together during the handling of B_ℓ.) So, during the handling of B_m, the permutations of $\mathbf{r}$ and $\mathbf{r}'$ that appear in B_m do not need to be paired with any other rows; these rows already have been assigned partners. As explained above, we know that the appropriate permutation of one

of $\mathbf{r}$ and $\mathbf{r}'$ must appear in X_m, while the appropriate permutation of the other must appear in Y_m. So, the number of unpaired rows in X_m and the number of unpaired rows in Y_m both decrease by 1. Thus, the number of unpaired rows in X_m is equal to the number of unpaired rows in Y_m. Therefore, since any row in X_m can be paired with any row in Y_m, it is still possible to produce zero-masking pairs between the unpaired rows in X_m and the unpaired rows in Y_m.

We now repeat this process by performing another column swap to move A into a new B_p form, where $p \neq \ell$, $p \neq m$. Again, any row in X_p that has already been paired during the handling of a previous B_s form will imply there is another row in Y_p that has also already been paired. Again it is implied that there will always be an equal number of unpaired rows in X_p as unpaired rows in Y_p.

We continue to perform column swaps as necessary until we have handled all B_ℓ forms, for $\ell = 1, 2, \ldots, u$, where u is the number of variables. (It may be possible to terminate this algorithm without examining every B_ℓ form, due to the overlap between B_ℓ's, but here we give the general algorithm.) Since every row is contained in some B_ℓ form (in fact, each row appears in $n + 1$ B_ℓ forms), we are guaranteed that each row will be paired.

Since we have shown it is possible to partition the rows of A into mutually exclusive zero-masked pairs, it follows directly that we can produce a zero-masking row companion matrix for A. Thus, we have shown that given any maximum rate GCOD A with $2n$ columns and $\kappa \binom{2n}{n-1}$ rows, where $\kappa \in \{1, 2\}$ is chosen so that $\kappa \binom{2n}{n-1}$ is even, we can find a zero-masking row companion matrix B such that $A + \mathbf{j}B$ is a zero-free QOD with $2n$ columns. Since deleting any column of $A + \mathbf{j}B$ produces a QOD with $2n - 1$ columns, we can use this construction to produce a zero-free QOD with any number of columns. □

Theorem 3 is important because it allows us to use a maximum rate GCOD with $2n$ columns to generate a zero-free QOD with $2n$ columns, or with $2n - 1$ columns after a column deletion. Therefore, we have a general algorithm for producing zero-free QODs for any number of columns.

We focused here on obtaining zero-free designs over the quaternion domain because zero-free designs over the complex domain have been shown to be important in the application of space-time block coding [16]. It remains to show if the zero-free nature of the proposed QODs will be exploitable in future applications as space-time-polarization block codes. We conjecture that we will need a stronger condition such that all entries are simultaneously non-zero in both polarization planes. This is stronger than the present condition that no entry is identically equal to zero, which allows entries to be zero in one polarization plane.

4.3. Redundant Quaternion Orthogonal Designs

In this section, we provide a construction for QODs on quaternion variables, as opposed to QODs on complex variables. Previously, it has been more difficult to obtain QODs on quaternion variables, due to the added dimensions in the entries and the noncommutativity of the quaternions [2]. The following theorem provides the first known direct construction to yield QODs on quaternion variables for any number of columns.

Theorem 4. *Let A be a maximum rate complex orthogonal design with $2n$ columns and $2\binom{2n}{n-1}$ rows. Then it is possible to form a row companion matrix B, so that $A + \mathbf{j}B$ is a quaternion orthogonal design on quaternion variables.*

Proof. It has been shown previously that a maximum rate GCOD with $2n$ columns and $2\binom{2n}{n-1}$ rows has each possible pattern of $n-1$ zeros appearing in exactly two rows of A [15]. Therefore, we can form mutually exclusive pairs by matching together rows whose zero patterns are identical. The matrix formed via the combination $A + \mathbf{j}B$ will have a zero entry in each place that A had a zero entry. The importance is that given the (ℓ, m) entry of A as the complex variable $z = a + b\mathbf{i}$, and given the (ℓ, m) entry of B as the complex variable $z' = c + d\mathbf{i}$, where a, b, c, d are real variables, we have the (ℓ, m) entry of $A + \mathbf{j}B$ as $a + b\mathbf{i} + c\mathbf{j} - d\mathbf{k}$, which is a full quaternion variable. Hence, $A + \mathbf{j}B$ is a quaternion orthogonal design on quaternion variables, as opposed to a quaternion orthogonal design on complex variables. Since deleting any column of $A + \mathbf{j}B$ produces a QOD with $2n - 1$ columns, we can use this construction to produce a QOD with any number of columns. □

We note that the construction in Theorem 4 can be readily implemented, as Liang's algorithm provides the necessary maximum rate GCODs of size $2\binom{2n}{n-1} \times 2n$ for all n [5].

Example 6. Let A be as in Example 2. Then, form a row companion matrix B' by matching together rows with the same zero patterns:

$$
B' = \begin{bmatrix}
z_6^* & 0 & -z_5^* & z_4^* \\
0 & z_6^* & z_3^* & -z_2^* \\
z_5 & -z_3 & z_6 & 0 \\
-z_4 & z_2 & 0 & z_6 \\
\hline
z_1 & 0 & z_2 & z_3 \\
0 & z_1 & z_4 & z_5 \\
-z_2^* & -z_4^* & z_1^* & 0 \\
-z_3^* & -z_5^* & 0 & z_1^*
\end{bmatrix}.
$$

Then, we have

$$
A + \mathbf{j}B' = \begin{bmatrix}
z_1 + \mathbf{j}z_6^* & 0 & z_2 - \mathbf{j}z_5^* & z_3 + \mathbf{j}z_4^* \\
0 & z_1 + \mathbf{j}z_6^* & z_4 + \mathbf{j}z_3^* & z_5 - \mathbf{j}z_2^* \\
-z_2^* + \mathbf{j}z_5 & -z_4^* - \mathbf{j}z_3 & z_1^* + \mathbf{j}z_6 & 0 \\
-z_3^* - \mathbf{j}z_4 & -z_5^* + \mathbf{j}z_2 & 0 & z_1^* + \mathbf{j}z_6 \\
\hline
z_6^* + \mathbf{j}z_1 & 0 & -z_5^* + \mathbf{j}z_2 & z_4^* + \mathbf{j}z_3 \\
0 & z_6^* + \mathbf{j}z_1 & z_3^* + \mathbf{j}z_4 & -z_2^* + \mathbf{j}z_5 \\
z_5 - \mathbf{j}z_2^* & -z_3 - \mathbf{j}z_4^* & z_6 + \mathbf{j}z_1^* & 0 \\
-z_4 - \mathbf{j}z_3^* & z_2 - \mathbf{j}z_5^* & 0 & z_6 + \mathbf{j}z_1^*
\end{bmatrix}.
$$

The importance of this construction is that each non-zero entry of the QOD is a full quaternion variable of the form $\mathbf{a} = a_1 + a_2\mathbf{i} + a_3\mathbf{j} + a_4\mathbf{k}$, where the a_h are *nonzero* real variables. It has previously been difficult to generate such QODs. We call the QODs formed via this construction technique *redundant* QODs due to the regular redundancy present among their entries. For example, the $(1, 1), (2, 2), (3, 3), (4, 4), (5, 1), (6, 2), (7, 3)$, and $(8, 4)$ entries of $A + \mathbf{j}B'$ form a "redundant family" of quaternion variables. Also, the $(1, 3), (2, 4), (3, 1), (4, 2), (5, 3), (6, 4), (7, 1), (8, 2)$ entries form a redundant family, and the $(1, 4), (2, 3), (3, 2), (4, 1), (5, 4), (6, 3), (7, 2), (8, 1)$ entries form a redundant family. This is equivalent to saying that the design is defined over three

independent quaternion variables, with each independent quaternion variable, up to conjugation and left and/or right multiplications by entries from $\mathcal{Q}$, appearing exactly twice per column.

In general, when the construction technique described in the proof of Theorem 4 is applied to appropriate row companion complex orthogonal designs on u complex variables, there will be $\frac{u}{2}$ redundant families of quaternion variables, or equivalently, $\frac{u}{2}$ independent quaternion variables. The regular redundancy is a consequence of the structure imposed by the B_ℓ submatrices of A and B. Practically, this regular redundancy may be exploitable in future applications of these designs as space–time-polarization codes. The redundant families, or subfamilies thereof, may be used to introduce redundancy for use in error control. Future implementations may determine if the proposed zero-free QODs or the proposed redundant QODs are preferred over general QODs.

5. Conclusions

In this article, we focused on using complex orthogonal designs to build quaternion orthogonal designs. Through this work, we have added to the nascent library of QODs and provided insight into the structure of these mathematical designs.

We presented the column companion $A + B\mathbf{j}$ construction and showed that this construction technique is valid whenever A and B are chosen so that $A^H B$ and $B^H A$ are real and equal. We provided a simple method for constructing column companion matrices A and B such that $A^H B = B^H A$ is real. This construction technique generates an $r \times 2n$ QOD on complex variables from any $r \times 2n$ generalized complex orthogonal design, which can themselves be generated for any choice of number of columns using well-known algorithms [5,17].

We presented the row companion $A + \mathbf{j}B$ construction and showed that this construction technique is valid whenever A and B are chosen so that $A^T B$ is symmetric. We used row companion matrices as a way to build B from A in order to guarantee that $A^T B$ is symmetric.

Using the latter construction technique, we showed that given a maximum rate generalized complex orthogonal design A with $2n$ columns that achieves either the minimum decoding delay or twice the minimum decoding delay (depending on n), it is possible to find a zero-masking row companion matrix B such that $A + \mathbf{j}B$ is a zero-free QOD on complex variables. Since deleting a column from any orthogonal design produces an orthogonal design, it is possible to use the proposed construction to build zero-free QODs on complex variables for any number of columns.

We also used this construction to produce QODs on quaternion variables. In particular, we showed that given a maximum rate generalized complex orthogonal design A with $2n$ columns that achieves twice the minimum decoding delay, it is possible to find a row companion matrix B such that $A + \mathbf{j}B$ has each entry as either 0 or a full quaternion variable. We called the resulting QODs *redundant* due to a regular redundancy appearing in the nonzero entries.

All constructions in this article require only readily available generalized complex orthogonal designs. Due to existing algorithms for generating maximum rate, minimum decoding delay designs [17] and maximum rate, twice minimum delay designs [5,17], the proposed infinite families of QODs are easily obtainable.

We note that all of the presented constructions can be modified to apply to real companion matrices in order to build complex orthogonal designs using combinations of the form $A + \mathbf{i}B$. However, other existing algorithms for maximum rate, minimum delay complex orthogonal designs are preferable in applications due to the higher attainable rates [5,17]. It remains to determine the maximum attainable rates using the quaternion domain.

We hope that Corollaries 1 and 2 stimulate further work on complex amicable and amicable-like designs. Our future work will also include a study of the optimal rates and delay of QODs and an exploration of the practical implementation of these designs as space–time-polarization block codes. We will investigate if the different attributes of our designs (*e.g.*, zero-free, redundant) can be exploited to improve the performance of the proposed codes.

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