Novel Classes of Minimal Delay and Low PAPR Rate $\frac{1}{2}$ Complex Orthogonal Designs

Sarah Spence Adams, *Member, IEEE*, James Davis, Nathaniel Karst, Mathav Kishore Murugan, Bryce Lee, Matthew Crawford, and Caitlin Greeley

***Abstract*—Complex orthogonal designs (CODs) of rate 1/2 have been considered recently for use in analog transmissions and as an alternative to maximum rate CODs due to the savings in decoding delay as the number of antennas increases. While algorithms have been developed to show that an upper bound on the minimum decoding delay for rate 1/2 CODs with $n = 2m - 1$ or $n = 2m$ columns is $\nu(n) = 2^{m-1}$ or $\nu(n) = 2^m$, depending on the parity of n modulo 8, it remains open to determine the exact minimum delay. This paper shows that this bound $\nu(n)$ is also a lower bound on minimum decoding delay for a major class of rate 1/2 CODs, named balanced complex orthogonal designs (BCODs), and that this is the exact minimum decoding delay for most BCODs. These rate 1/2 codes are conjugation-separated and thus permit a linearized description of the transceiver signal. BCODs also display other combinatorial properties that are expected to be useful in implementation, such as having no linear processing. An elegant construction is provided for a class of rate 1/2 CODs that have no zero entries, effectively no irrational coefficients, no linear processing, and have each variable appearing exactly twice per column. The resulting codes meet the aforementioned bound on decoding delay in most cases. This class of CODs will be useful in practice due to their low peak-to-average power ratio (PAPR) and other desirable properties.**

***Index Terms*—Complex orthogonal design, minimum decoding delay, multiple-input-multiple-output, PAPR, space-time block code.**

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S. S. Adams is with Franklin W. Olin College of Engineering, Olin Way, Needham, MA 02492-1245 (e-mail: sarah.adams@olin.edu).

J. Davis is with the University of Richmond, Richmond, VA 23173 USA (e-mail: jdavis@richmond.edu).

N. Karst is with Cornell University, Ithaca, NY 14850 USA (e-mail: nathankarst@gmail.com).

K. M. Murugan was with the Indian Institute of Technology, Kharagpur. He is now with Cornell University, Ithaca, NY (e-mail: mathavkishore@gmail.com).

B. Lee was with Franklin W. Olin College of Engineering, Needham, MA 02492-1245. He is now with Virginia Tech, Blacksburg, VA 24061 USA. (e-mail: bryce.kt.lee@gmail.com).

M. Crawford was with Franklin W. Olin College of Engineering, Olin Way, Needham, MA 02492-1245. He is now with Athenahealth, Inc., Watertown, MA 02472-2785 USA (e-mail: matthew.crawford@alumni.olin.edu).

C. Greeley was with Franklin W. Olin College of Engineering, Olin Way, Needham, MA 02492-1245. She is now with Oliver Wyman, New York, NY 10036-2726 USA (e-mail: cgreeley511@gmail.com).

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I. Motivation

A. Background

COMPLEX orthogonal designs and their various generalizations have been defined in a variety of ways [1]–[7]. In this paper, we use a classical definition and say that an (r, n, k) *complex orthogonal design (COD) of type* $(\alpha_1, \ldots, \alpha_k)$ is an $r \times n$ matrix $\mathbf{G}$ with complex entries from $\{0, \pm z_1, \ldots, \pm z_k, \pm z_1^*, \ldots, \pm z_k^*\}$ such that $\mathbf{G}^H\mathbf{G} = \sum_{i=1}^{k} \alpha_i |z_i|^2 \mathbf{I}_n$, where H is the Hermitian transpose and $\mathbf{I}_n$ is the $n \times n$ identity matrix. If entries are permitted to be complex linear combinations of the variables and/or their conjugates, then we say that the COD is with linear processing (LP) [8]. Unless otherwise specified, the CODs in this paper are of type $(1, 1, \ldots, 1)$ with no LP. This is the major type of combinatorial designs considered in many papers, including Liang's seminal publication concerning the maximum rate of such designs [9]. We note that recently Das and Rajan have used the term "scaled COD" to refer to a type $(\alpha_1, \ldots, \alpha_k)$ COD where $\alpha_i \in \{1, 2\}$ [10]. We will say that an (r, n, k) *real orthogonal design (ROD)* is an $r \times n$ matrix $\mathbf{R}$ with real entries from $\{0, \pm x_1, \ldots, \pm x_k\}$ such that $\mathbf{R}^T\mathbf{R} = \sum_{l=1}^{k} x_l^2 \mathbf{I}_n$, where T is the matrix transpose and $\mathbf{I}_n$ is the $n \times n$ identity matrix.

There are several characteristics of a COD that should be considered when choosing a COD for practical application. The design considerations include:

C1: Rate;
C2: Decoding delay;
C3: Transceiver signal linearization;
C4: Peak-to-average power ratio (PAPR);
C5: Power balance;
C6: Irrational coefficients;
C7: Linear processing.

Below, we briefly explain the importance of each of these design considerations and give some pointers to prior work in the respective areas. Yuen *et al.* have also recently discussed optimal characteristics of CODs, including with respect to some of the above considerations [11].

C1 and C2: Liang showed that the maximum rate (e.g., maximum ratio of number of variables to number of rows) for a COD with $2m - 1$ or $2m$ columns is $\frac{m+1}{2m}$ [9]. Adams *et al.* subsequently showed that for a maximum rate COD with $2m - 1$ or $2m$ columns, a lower bound on minimum decoding delay (e.g., number of rows) is $\binom{2m}{m-1}$ [12]; furthermore, this bound on decoding delay is achievable when the number of antennas is congruent to 0, 1, or 3 modulo 4 [12]. Adams *et al.* went on to prove that when the number of antennas is congruent to 2 modulo 4, the best achievable decoding delay is $2\binom{2m}{m-1}$ [13]. This delay grows quickly as the number of antennas increases, while the rate at this limit approaches 1/2. Thus, interest in rate 1/2 CODs has grown, and an important question is to determine the minimal decoding delay of rate 1/2 CODs. Rate 1/2 CODs have also received attention due to their proposed use in analog transmissions [14].

C3: When implementing a COD using iterative decoding, it is preferred to use a COD that allows for a linearized description of the transceiver signal [15]–[18]. Su *et al.* recently determined the conditions under which a COD can achieve this transceiver signal linearization [17]; they showed that this linearization is achieved if the COD is such that the nonzero entries in any given row are either all conjugated ("a conjugated row") or all nonconjugated ("a nonconjugated row") [17]. We will call this property *conjugation-separation*.

C4: A low peak-to-average power ratio (PAPR) mitigates several potential implementation difficulties and can be achieved by reducing the number of zeros in the COD. Fewer zero entries reduces the need to switch on and off antennas which is known to complicate implementation (see [11] and the references therein). Zero reduction or elimination has been explored by many authors (e.g., [11], [19]–[24]). Multiplying a COD by a unitary matrix is an easy method for producing a COD with no zero entries. However, this can result in increased signaling complexity, for example by introducing complex linear combinations of variables in the entries of the design (linear processing). As Das and Rajan explain [20], finding suitable unitary matrices that do not result in increased signaling complexity is difficult, though successful pre- and postmultiplications can be seen in their recent works [20], [21]. To avoid a potential increase in signaling complexity or to avoid the nontrivial task of finding suitable unitary matrices, it is preferable to devise a simple construction that immediately provides a COD with no zero entries. Direct construction of such zero-free CODs is also an interesting problem from a mathematical perspective.

C5: A COD is said to be power-balanced if each variable appears the same number of times in each column, that is if the COD has type $(\alpha, \ldots, \alpha)$. Yuen *et al.* show that in this case the transmitted symbols will have equal transmission power, thus eliminating the need for power normalizations [11].

C6: Another consideration is whether a COD includes any irrational coefficients on any of the entries. First, we note that a COD $\mathbf{G}$ of type $(\alpha, \ldots, \alpha)$ satisfies $\mathbf{G}^H\mathbf{G} = \sum_{i=1}^{k} \alpha|z_i|^2\mathbf{I}_n$, and such a COD could be scaled by an overall factor of $\frac{1}{\sqrt{\alpha}}$ in order to satisfy the alternative common definition of a COD that is restricted to type $(1, \ldots, 1)$ and therefore satisfies $\mathbf{G}^H\mathbf{G} = \sum_{i=1}^{k} |z_i|^2\mathbf{I}_n$. In this latter formulation, since the overall matrix is scaled by the same possibly irrational coefficient, we can effectively ignore this coefficient and say that the COD has effectively no irrational coefficients. Having (effectively) no irrational coefficients is desirable in practice, since their absence eliminates the inconvenience and inaccuracy of floating point multiplication and simplifies the hardware implementation [11], [25].

C7: Finally, we note that a COD with no linear processing (LP) is desired, because the presence of complex linear combinations of variables (including the use of irrational coefficients) increases signaling and decoding complexity (see, for example, [20]).

While we cannot simultaneously optimize every design consideration except in the case of the 2×2 Alamouti code [26], we strive to achieve a good balance of rate and delay (recalling that delay considerations are a motivation for studying rate 1/2 as opposed to maximum rate CODs) while optimizing as many of the other considerations as possible. In many cases, specific system requirements will guide which considerations are more or less important. Additional considerations may also be important for certain applications. Several authors have taken diverse approaches towards building optimal CODs, and we discuss some of these significant efforts below.

B. Summary of Prior Results

Tarokh *et al.* used a rate 1 $(\nu(n), n, \nu(n))$ ROD to obtain a rate 1/2 $(2\nu(n), n, \nu(n))$ COD with decoding delay $2\nu(n)$, where $\nu(n)$ is related to the Hurwitz-Radon number of order n as defined below in Section II [8]. These rate 1/2 CODs perform well with respect to characteristics C3–C7 above, and they are obtained via a simple algorithm. However, we will see that their delay can be reduced by 50%.

Subsequently, Liang remarked (without proof) that any rate 1 $(\nu(n), n, \nu(n))$ ROD "is itself" a rate 1/2 $(\nu(n), n, \frac{1}{2}\nu(n))$ COD of the same delay by pairing real variables appropriately [9]. Although the details of such a correspondence have not been published, rate 1/2 CODs generated using such a rule achieve a delay of $\nu(n)$, which is presumed to be the minimum delay as discussed in more detail in later sections of this paper. Hence, these rate 1/2 CODs are presumed optimal with respect to consideration C2; they also perform well with respect to considerations C4 and C5. However, they do not achieve transceiver signal linearization (C3), and they require LP (C7).

Recently, Das and Rajan [21] have proposed a class of rate 1/2 CODs that achieve the same presumed optimal delay as achieved by Liang, and they showed that for the case of nine columns, their COD is indeed of minimum delay among 1/2 rate CODs that allow LP. These CODs themselves do not achieve C3–C7, and they are obtained through an iterative algorithm. However, the authors observe that the presence of columns with complementary zero patterns in the resulting CODs allows for the design of a simple unitary matrix whose postmultiplication with their original CODs produces type $(2, \ldots, 2)$ zero-free CODs with the same delay as the originals. Hence, these new designs perform well with respect to design considerations C4–C7, though they do not admit transceiver linearization (C3).

Through these works, $\nu(n)$ has been established as an upper bound on the minimum decoding delay for rate 1/2 CODs. It has only been proved to be the exact minimum in the case of nine columns [21].

C. Overview of Paper

In this paper, we introduce a large class of rate 1/2 *balanced complex orthogonal designs* (BCODs) (see Section III). These BCODs are generated using a modification of the well-known Liang algorithm for generating maximum rate CODs [9]. This class of CODs is important because they are the only known type

$(1,\ldots,1)$ rate 1/2 CODs that simultaneously achieve transceiver signal linearization, are power-balanced, have no irrational coefficients, have no linear processing, and, for most numbers of columns, achieve the conjectured lower bound on decoding delay (i.e., they perform well with respect to design considerations C3 and C5-C7, as well as C2 for most numbers of columns).

One of the major contributions of this paper is determining the exact minimum decoding delay for most BCODs (see Section IV). This is the first known proof concerning the minimum decoding delay for a large class of rate 1/2 CODs. Previously, the only proof regarding the minimum decoding delay of rate 1/2 CODs was limited to the case of nine columns [21]. Our work represents a significant step towards resolving the minimum decoding delay question for all rate 1/2 CODs.

In Section V, we also present an elegant algorithm to construct rate 1/2 CODs of type $(2,\ldots,2)$. These CODs achieve the conjectured lower bound on delay for most numbers of columns, thus performing well with respect to design consideration C2. Furthermore, for any number of columns, these designs perform well with respect to C4-C7. This is the first known class of rate 1/2 CODs that achieve low delay, no zeros, power balance, no irrational coefficients, no linear processing for any number of columns, and are obtained through a holistic algebraic approach avoiding algorithms that are iterative, case-based, and/or reliant on pre- or postmultiplication by unitary matrices. It is interesting to note that Das and Rajan [21] generate well-performing type $(2,\ldots,2)$ rate 1/2 CODs through a seemingly completely different approach, yet a close comparison of their and our designs show that the different approaches can actually be recast to appear more similar. It is possible to recast our holistic algebraic approach into one that uses premultipication by unitary matrices. We omit these details as our algebraic approach has advantages such as allowing for mathematical proofs regarding certain properties of the designs. We also remark that our proposed codes satisfy the guidelines recently proposed by Yuen *et al.* for practical orthogonal space-time block codes [11].

We note that our BCOD construction of Sections III and IV supports transceiver signal linearization (C3), but fails to have low PAPR (C4), while the opposite is true for our type $(2,\ldots,2)$ construction of Section V. This, along with the works cited in the previous subsection, highlights the inherent conflict between different implementation issues and gives a sense of the mathematical compromises that must be made when constructing a COD.

II. Preliminaries

In this section, we develop some algebraic notation and review some useful results.

Throughout this paper, we use the following standard equivalence operations which can be performed on any COD: 1) rearrange the order in which the rows appear in the matrix ("row rearrangements"); 2) rearrange the order in which the columns appear in the matrix ("column rearrangements"); 3) conjugate and/or negate all instances of certain variables; and 4) multiply any row and/or column by -1.

For example, given a COD $\mathbf{G}$, we can perform a series of row rearrangements represented by the function ϕ to obtain a COD $\phi(\mathbf{G})$ whose rows are the rows of $\mathbf{G}$ simply appearing in a different order. We say that $\mathbf{G}$ and $\phi(\mathbf{G})$ are equivalent designs, and throughout this paper we will consider various equivalent versions of a given design. Through a minor abuse of notation, we often refer to any $\phi(\mathbf{G})$, where ϕ represents any combination of equivalence operations, still as $\mathbf{G}$.

For any COD $\mathbf{G}$ with $2m-1$ or $2m$ columns, and for any fixed $z_i, 1 \leq i \leq k$, Liang showed that the orthogonality constraint implies that it is possible to transform $\mathbf{G}$ through equivalence operations so that the following submatrix $\mathbf{B}_i$, which contains all appearances of the variable z_i, appears within the top $2m-1$ or $2m$ rows, respectively, [9]

$$\mathbf{B}_i = \begin{pmatrix} z_i\mathbf{I} & \mathbf{M}_i \\ -\mathbf{M}_i^H & z_i^*\mathbf{I} \end{pmatrix}.$$

We generalize the notion of $\mathbf{G}$ displaying the $\mathbf{B}_i$ submatrix to say that $\mathbf{G}$ is in $\mathbf{B}_i$ *form* if the rows of the submatrix $\mathbf{B}_i$ appear in some order within $\mathbf{G}$, up to the conjugation and sign of their entries. This notion of $\mathbf{B}_i$ form first appeared in [12].

In the study of rate 1/2 CODs with $2m$ columns, it is natural to consider examples wherein the submatrices $\mathbf{M}_i$ and $-\mathbf{M}_i^H$ each have exactly one zero per row and one zero per column. In these examples, it is possible to rearrange the rows/columns of $\mathbf{G}$ so that $\mathbf{M}_i$, hence $-\mathbf{M}_i^H$, have zero entries along the diagonal. Generalizing Liang's work on the $\mathbf{B}_i$ submatrix, we define the *strong* $\mathbf{B}_i$ submatrix to be the usual $\mathbf{B}_i$ submatrix with the additional requirement that the $\mathbf{M}_i$ submatrix has 0 entries along the diagonal. We say that such a COD $\mathbf{G}$ is in *strong* $\mathbf{B}_i$ *form* if the rows of the strong $\mathbf{B}_i$ submatrix appear in some order within the rows of $\mathbf{G}$, up to the conjugation and sign of their entries.

The Hurwitz-Radon numbers $\rho(n)$ are important in the study of collections of mutually orthogonal matrices and are defined as follows: If $n = 2^a(2b+1)$ and $a = 4c+d$, where a, b, c, and d are integers with $0 \leq d < 4$, then $\rho(n) = 8c + 2^d$ [27]–[29]. Liang [9] reviews that this is equivalent to the following:

$$\rho(n) = \begin{cases} 2a+1, & \text{if } a \equiv 0 \text{ modulo } 4 \\ 2a, & \text{if } a \equiv 1 \text{ modulo } 4 \\ 2a, & \text{if } a \equiv 2 \text{ modulo } 4 \\ 2a+2, & \text{if } a \equiv 3 \text{ modulo } 4. \end{cases}$$

Then, following Liang's notation, we can define $\nu(n) = \min\{p|\rho(p) \geq n\}$. Finally, writing $\nu(n) = 2^{\delta(n)}$, we have the equation at the bottom of the page [9]. The δ function

$$\begin{aligned}\delta(n) &= \min\{m|\rho(2^m) \geq n\} \\ &= \begin{cases} 4t, & \text{if } n = 8t+1 \\ 4t+1, & \text{if } n = 8t+2 \\ 4t+2, & \text{if } n = 8t+3, \text{ or } n = 8t+4 \\ 4t+3, & \text{if } n = 8t+5, n = 8t+6, n = 8t+7, \text{ or } n = 8t+8. \end{cases}\end{aligned}$$

then simplifies as shown in the equation at the bottom of the page.

Recently, algorithms have been developed to show that $\nu(n)$, which simplifies to 2^{m-1} or 2^m depending on the parity of $n = 2m-1$ or $2m$ modulo 8, is an upper bound on the decoding delay for rate 1/2 CODs with n columns [10]. In Section IV, we prove that $\nu(n)$ is also a tight lower bound for a major class of rate 1/2 CODs.

III. Balanced Complex Orthogonal Designs

In this section, we present an important class of rate 1/2 CODs named *balanced complex orthogonal designs (BCODs)*. We begin by describing a modification of Liang's well-known algorithm for generating maximum rate CODs [9]; this *modified-Liang algorithm* generates rate 1/2 CODs for any number of columns. It is convenient in this modification to directly generate $(2^m, 2m, 2^{m-1})$ CODs, and then use column deletion to obtain $(2^m, 2m-1, 2^{m-1})$ CODs, so we will only describe the algorithm for the case with $2m$ columns.

The four steps of our modification follow closely the four steps of Liang's algorithm [9], with the only significant changes occurring in Steps 1 and 3. Step 1 of Liang's algorithm initializes the scaled identity submatrices of $\mathbf{B}_1$ while leaving $\mathbf{M}_1$ and $-\mathbf{M}_1^H$ empty. Step 1 of our modified-Liang algorithm completes the same assignments with the additional placement of zeros along the main diagonals of $\mathbf{M}_1$ and $-\mathbf{M}_1^H$.

Step 3 of Liang's algorithm fills the empty entries of a $\mathbf{B}_i$ submatrix with new complex variables, each appearing once as z_j and $-z_j^*$ within $\mathbf{M}_i$ and $-\mathbf{M}_i^H$ submatrices, respectively. In contrast, Step 3 of our modified-Liang algorithm fills the empty entries of a $\mathbf{B}_i$ submatrix with new complex variables each appearing once as both z_j and $-z_j$ in $\mathbf{M}_i$ and once as both z_j^* and $-z_j^*$ in $-\mathbf{M}_i^H$ such that if the strong $\mathbf{B}_i$ submatrix appears in $\mathbf{G}$ for any i, then the $\mathbf{M}_i$ submatrix is skew-symmetric with zeros along the diagonal.

The details of the algorithm are left to the reader, who can follow Liang's presentation with the above changes to obtain the desired rate 1/2 CODs. Furthermore, we can mimic Liang's proofs concerning the number of variables in his resulting CODs to prove that our modified algorithm gives a delay of 2^m, thus producing $(2^m, 2m, 2^{m-1})$ CODs. Then, by deleting any column, we obtain $(2^m, 2m-1, 2^{m-1})$ CODs. Thus, the modified-Liang algorithm (with the deletion of columns as necessary) confirms the upper bound on decoding delay of $\nu(n) = 2^m$ when $n = 2m-1$ or $2m$ is congruent to 2, 3, 4, 5, or 6 modulo 8 (while the delay is $2\nu(n)$ when n is congruent to 0, 1, or 7 modulo 8).

Although other algorithms have been proposed recently that establish this upper bound of $\nu(n)$ for all congruence classes of n [10], the modified-Liang algorithm given here has some important advantages. As we will discuss in more detail below, our modified-Liang algorithm generates the only known type $(1,\ldots,1)$ rate 1/2 CODs that perform well with respect to all of design considerations C3, C5, C6, and C7 for all numbers of columns and that additionally perform well with respect to C2 for most numbers of columns.

We now incorporate some of the algebraic and combinatorial properties of the CODs generated by the modified-Liang algorithm in following definition:

Definition 3.1: A COD $\mathbf{G}$ with $n = 2m$ columns is a *balanced complex orthogonal design (BCOD)* if it satisfies the following conditions.

1) For each $i = 1, 2, \ldots, k, z_i$ and z_i^* each appear m times (up to sign).
2) Every row of $\mathbf{G}$ has exactly m zero and m nonzero entries.
3) $\mathbf{G}$ is conjugation-separated.
4) For each $i = 1, 2, \ldots, k$, the $\mathbf{M}_i$ submatrix of the strong $\mathbf{B}_i$ submatrix is skew-symmetric.

It follows that any BCOD (generated from any algorithm, not restricted to our modified-Liang algorithm) is a rate 1/2 $(2k, 2m, k)$ COD for some positive integers k, m.

By condition 1), for any BCOD, the $\mathbf{M}_i$ submatrix of any $\mathbf{B}_i$ submatrix is size $m \times m$. Furthermore, condition 2) implies that $\mathbf{M}_i$ (resp., $-\mathbf{M}_i^H$) has exactly one zero per row, which implies that $-\mathbf{M}_i^H$ (resp., $\mathbf{M}_i$) has exactly one zero per column. Thus, $\mathbf{M}_i$ and $-\mathbf{M}_i^H$ each have exactly one zero per row and one zero per column. So it is possible to rearrange the rows/columns of $\mathbf{G}$ so that $\mathbf{M}_i$, hence $-\mathbf{M}_i^H$, have zero entries along the diagonal, thereby obtaining the the *strong* $\mathbf{B}_i$ *submatrix* defined in Section II. This regular structure with predictable zero patterns may be exploitable in applications.

All $(2k, 2m, k)$ codes generated by the modified-Liang algorithm have the additional property that the $\mathbf{M}_i$ submatrices of all strong $\mathbf{B}_i$ submatrices are skew-symmetric, so that $\mathbf{M}_i^T = -\mathbf{M}_i$ for all $1 \le i \le k$. We call this the *skew-symmetric submatrix property (SSSP)*, which we have observed to be a result of conditions 1)–3) in Definition 3.1, regardless of the algorithm used to generate the design.

When evaluating BCODs against the design considerations outlined in Section I-A, we see that BCODs are desirable codes. Condition 3) implies that these codes can achieve transceiver signal linearization [17], which is design consideration C3. It also follows directly from their definition that that BCODs perform well with respect to considerations C5–C7, as they are power-balanced and have no irrational coefficients or linear processing. Furthermore, these CODs have an even number of columns, which is preferable in practice: An analysis of outage probability concluded that codes with an even number of antennas outperform comparable codes with an odd number (one more or one fewer) of antennas [30], and this analysis was extended to include other performance measures such as the mean-square error and the bit-error rate [31]. We conclude that the proposed BCODs are expected to be among the most useful type $(1,\ldots,1)$ rate 1/2 CODs for practical implementations,

$$\delta(n) = \begin{cases} m, & \text{if } n = 2m-1 \text{ or } 2m, n \equiv 2,3,4,5, \text{ or } 6 \text{ modulo } 8 \\ m-1, & \text{if } n = 2m-1 \text{ or } 2m, n \equiv 0,1, \text{ or } 7 \text{ modulo } 8. \end{cases}$$

especially when the number of columns n is congruent to 2, 4, or 6 modulo 8 so that they achieve a decoding delay of $\nu(n)$ (the conjectured lower bound on minimum decoding delay). Indeed, in these cases, they are the only known type $(1,\ldots,1)$ rate 1/2 CODs that perform well with respect to C2, C3, and C5–C7. These BCODs are easily generated via the modified-Liang algorithm, and they are combinatorially rich.

In the following Section IV, we will show that the upper bound of $\nu(n)$ on the decoding delay of rate 1/2 CODs is also a lower bound for BCODs and is the exact minimum decoding delay for most BCODs. We will also use these BCODs in Section V as building blocks for low PAPR rate 1/2 CODs.

IV. The Minimum Decoding Delay of BCODs

In this section, we prove that the upper bound of $\nu(n)$ on the minimum decoding delay of rate 1/2 CODs with n columns is also a lower bound on minimum decoding delay for BCODs. In most cases, this lower bound is tight and achievable, thus proving the exact minimum decoding delay of most BCODs. We begin by reviewing some notation introduced in [32], slightly modified here for our specialized application.

Definition 4.1: Let $\mathbf{A}$ be a COD. We define a *row companion matrix* $\mathbf{B}$ by permuting the rows of $\mathbf{A}$ according to a permutation π with order 2, so that $\pi^2=1$ and π is fixed-point free.

As an example, note that given a $(2k,2m,k)$ BCOD $\mathbf{A}$, we can build a row companion matrix $\mathbf{B}$ using the order 2 permutation π with cycle notation $\pi=(12)(34)\cdots(2k-1,2k)$. Then, if $\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_{2k}$ denote the $2k$ rows of $\mathbf{A}$ in order, the rows of $\mathbf{B}$ are $\mathbf{r}_2,\mathbf{r}_1,\mathbf{r}_4,\mathbf{r}_3,\ldots,\mathbf{r}_{2k},\mathbf{r}_{2k-1}$ in order. If $\mathbf{A}_i$ denotes the ith column of $\mathbf{A}$ for all $i=1,2,\ldots,2m$, then $\pi(\mathbf{A}_i)$ represents the ith column of $\mathbf{B}$.

Definition 4.2: A COD $\mathbf{A}$ is *zero-maskable by rows* if the rows of $\mathbf{A}$ can be partitioned into pairs such that the two rows in a given pair are never both zero in the same column. If we form a row companion matrix $\mathbf{B}$ whose rows are, in order, the zero-masking partners of the rows of $\mathbf{A}$, in order, then $\mathbf{B}$ is called a *zero-masking row companion* for $\mathbf{A}$.

Note that if a COD $\mathbf{B}$ is a zero-masking row companion of a COD $\mathbf{A}$, then $\mathbf{A}+\mathbf{B}$ has no zero entries.

Lemma 4.3: Let $\mathbf{A}$ be a $(2k,2m,k)$ BCOD. Every row $\mathbf{r}$ of $\mathbf{A}$ has precisely one other *partner* row $\mathbf{r}'$ so that i) $\mathbf{r}$ and $\mathbf{r}'$ have complementary zero patterns; ii) $\mathbf{r}$ and $\mathbf{r}'$ have opposite conjugation; and iii) $\mathbf{r}$ and $\mathbf{r}'$ include the same variables.

Proof: Let $\mathbf{r}$ be any nonconjugated row of $\mathbf{A}$, and let z_s be any variable included in $\mathbf{r}$. Apply an appropriate series of equivalence operations, denoted by ϕ, to $\mathbf{A}$ so that $\phi(\mathbf{A})$ contains the strong $\mathbf{B}_s$ submatrix in its top $2m$ rows with row $\phi(\mathbf{r})$ as the first row of $\phi(\mathbf{A})$. By the definition of the strong $\mathbf{B}_s$ submatrix, rows 1 and $m+1$ have complementary zero patterns and opposite conjugation, and the SSSP implies that these rows include precisely the same variables. This last implication follows from the observation that $-\mathbf{M}_s^H=\mathbf{M}_s^*$. Applying equivalence operations to restore $\phi(\mathbf{A})$ to $\mathbf{A}$ while keeping track of the rows, we will have shown that row $\mathbf{r}$ in $\mathbf{A}$ has at least one partner row. If there were two partners for $\mathbf{r}$, then there would be two rows in strong $\mathbf{B}_s$ form both having identical zero patterns, a contradiction. Now choose any different nonconjugated row $\mathbf{t}$ of $\mathbf{A}$ and repeat the process. The partner row $\mathbf{t}'$ for $\mathbf{t}$ cannot already have been used since that would mean that $\mathbf{t}'$ had a previous row with complementary zero pattern and including the same variables, but that could only happen if $\mathbf{t}$ had already been chosen. Continue in this manner until all rows have been paired. ■

We now introduce an additional piece of machinery. Consider a $(2k,2m,k)$ BCOD $\mathbf{G}$. Define $\sigma_{\mathbf{G}}$ as a column vector of length $2k$ that is -1 in any row in which $\mathbf{G}$ is conjugated and $+1$ in any row in which $\mathbf{G}$ is nonconjugated. This is a well-defined construction, as $\mathbf{G}$ is conjugation-separated. Then, for any column vector $\mathbf{v}$ of length $2k$, let $\sigma_{\mathbf{G}}(\mathbf{v})$ denote the component-wise product of $\sigma_{\mathbf{G}}$ and $\mathbf{v}$. For example, given a column $\mathbf{G}_i$ of $\mathbf{G}, \sigma_{\mathbf{G}}(\mathbf{G}_i)$ is obtained from $\mathbf{G}_i$ by simply negating all conjugated variables in $\mathbf{G}_i$. Let $\sigma_{\mathbf{G}}(\mathbf{G})$ denote the matrix obtained by applying $\sigma_{\mathbf{G}}$ to all columns of $\mathbf{G}$. Then $\sigma_{\mathbf{G}}(\mathbf{G})$ is another valid BCOD, as negating rows is a valid equivalence operation for CODs and it does not affect the additional requirements of a BCOD.

As final preparation for our main result in this section, we recall that the minimum decoding delay for a rate 1 ROD with n columns is given by $\nu(n)$ [9], [28], [29]. In the proof of the following main result, we will show that the existence of a $(2k,2m,k)$ BCOD implies the existence of a rate 1 $(2k,2m,2k)$ ROD. Then, the tight lower bound of $\nu(n)$ on the delay of rate 1 RODs implies the same lower bound on the delay for $(2k,2m,k)$ BCODs.

Theorem 4.4: Let $\mathbf{A}$ be a $(2k,2m,k)$ BCOD. Then a lower bound on the minimum decoding delay of $\mathbf{A}$ is $\nu(2m)$.

Proof: Let $\mathbf{A}$ be an $(2k,2m,k)$ BCOD. Define its zero-masking row companion matrix $\mathbf{B}$ using the permutation Π defined on the rows of $\mathbf{A}$ by $\Pi(\mathbf{r})=\mathbf{r}'$, where $\mathbf{r}$ and $\mathbf{r}'$ are the partner rows guaranteed to exist by Lemma 4.3. Note that respective rows of $\mathbf{A}$ and $\mathbf{B}$ have complementary zero patterns, opposite conjugation, and the same variables; also, Π^2 is the identity and Π is fixed-point free. Then, since $\mathbf{B}$ is conjugation-separated, we can consider the BCOD $\sigma_{\mathbf{B}}(\mathbf{B})$, which is obtained by negating all of the conjugated rows of $\mathbf{B}$. For convenience, let $\mathbf{C}=\sigma_{\mathbf{B}}(\mathbf{B})$.

For all $1\le i\le k$, let ψ map the complex variables z_i in an arbitrary COD to real variables x_i while preserving sign, and let Ψ map the complex variables z_i in an arbitrary COD to real variables x_{i+k} while preserving sign. When a real variable x has a complex preimage z^* that is conjugated, the real variable will be denoted as x^*. We will now show that with $\mathbf{A}$ and $\mathbf{C}$ as above, $\psi(\mathbf{A})+\Psi(\mathbf{C})$ is a rate 1 $(2k,2m,2k)$ ROD.

We observe that $\psi(\mathbf{A})$ and $\Psi(\mathbf{C})$ have complementary zero patterns, so their sum will be a matrix with no zeros and no linear combinations of the real variables. So $\psi(\mathbf{A})+\Psi(\mathbf{C})$ is a $2k\times 2m$ matrix with entries from the $2k$ real variables $x_1,\ldots,x_{2k}$. It is clear that $\psi(\mathbf{A})$ and $\Psi(\mathbf{C})$ are individually $(2k,2m,k)$ RODs and hence $\psi(\mathbf{A})^T\psi(\mathbf{A})=(\Sigma_{i=1}^{k}|x_i|^2)\mathbf{I}_{2m}$ and $\Psi(\mathbf{C})^T\Psi(\mathbf{C})=(\Sigma_{i=k+1}^{2k}|x_i|^2)\mathbf{I}_{2m}$. Thus,

$$(\psi(\mathbf{A})+\Psi(\mathbf{C}))^T(\psi(\mathbf{A})+\Psi(\mathbf{C}))$$
$$=\left(\Sigma_{i=1}^{2k}|x_i|^2\right)\mathbf{I}_{2m}+\psi(\mathbf{A})^T\Psi(\mathbf{C})+\Psi(\mathbf{C})^T\psi(\mathbf{A}).$$

So it remains to prove that $\mathbf{D} = \psi(\mathbf{A})^T\Psi(\mathbf{C}) + \Psi(\mathbf{C})^T\psi(\mathbf{A}) = \mathbf{0}$.

If $\mathbf{G}$ is any matrix, then we use the notation $\mathbf{G}_i$ and $\mathbf{G}_{(i,j)}$ to respectively refer to the ith column and (i,j) entry of $\mathbf{G}$. Consider $\mathbf{D}_{(i,j)}$. If $i = j$, then $\mathbf{D}_{(i,j)} = 0$ since the columns of $\psi(\mathbf{A})$ and $\Psi(\mathbf{C})$ have complementary zero patterns. So we consider the case where $i \neq j$. Contributions to this (i,j) entry come from the inner product of $\psi(\mathbf{A})_i$ with $\Psi(\mathbf{C})_j$ and the inner product of $\Psi(\mathbf{C})_i$ with $\psi(\mathbf{A})_j$. We will show that given a contribution to these inner products from an arbitrary pth row, contributions are then implied from additional rows such that all contributions cancel to 0 as desired.

Consider an arbitrary row index $1 \leq p \leq 2k$, and assume that $\psi(\mathbf{A})_{(p,i)} = x_t$ and $\Psi(\mathbf{C})_{(p,j)} = x^*_{q+k}$, where x_t and x^*_{q+k} are real variables whose respective preimages under ψ and Ψ are complex variables z_t and $z^*_q, 1 \leq t, q \leq k$. The defining conditions of Π imply that exactly one of these entries is conjugated. We may assume without loss of generality that both entries are positive by negating columns (an equivalence operation) as necessary.

For simplicity, we will first consider the case where $t = q$, so that $\psi(\mathbf{A})_{(p,i)} = x_t$ and $\Psi(\mathbf{C})_{(p,j)} = x^*_{t+k}$. Then, Lemma 4.3 implies that the row of $\mathbf{A}$ with index p has a partner row with some index p' such that $\psi(\mathbf{A})_{(p',j)} = -x^*_t, \Psi(\mathbf{C})_{(p',i)} = x_{t+k}$, and $\psi(\mathbf{A})_{(p,j)} = \psi(\mathbf{A})_{(p',i)} = \Psi(\mathbf{C})_{(p,i)} = \Psi(\mathbf{C})_{(p',j)} = 0$. So rows with index p contribute the summand $(x_t)(x^*_{t+k})$ in the inner product of $\psi(\mathbf{A})_i$ with $\Psi(\mathbf{C})_j$ and a summand of 0 in the inner product of $\Psi(\mathbf{C})_i$ with $\psi(\mathbf{A})_j$; the rows with index p' contribute $(x_{t+k})(-x^*_t)$ in the inner product of $\Psi(\mathbf{C})_i$ with $\psi(\mathbf{A})_j$ and a summand of 0 in the inner product of $\psi(\mathbf{A})_i$ with $\Psi(\mathbf{C})_j$. Thus, the relevant contributions from rows with index p are canceled by the contributions from rows with index p', and we have $\mathbf{D} = \mathbf{0}$, as required.

Now consider the case where $t \neq q$. Then, as above, the rows with index p contribute $x_t x^*_{q+k}$ to $\mathbf{D}_{(i,j)}$. Lemma 4.3 implies that there exists some index p' such that $\psi(\mathbf{A})_{(p',j)} = -x^*_q$ and $\Psi(\mathbf{C})_{(p',i)} = x_{t+k}$. Thus, rows with index p' provide the summand $-x_{t+k}x^*_q$ within the inner product of $\psi(\mathbf{A})_j$ with $\Psi(\mathbf{C})_i$ and provide a summand of 0 within the inner product of $\psi(\mathbf{A})_i$ with $\Psi(\mathbf{C})_j$. Hence, the rows with indices p and p' provide a total contribution of $x_t x^*_{q+k} - x_{t+k}x^*_q$ in $\mathbf{D}_{(i,j)}$.

Note that there is a row with some index s such that $\psi(\mathbf{A})_{(s,j)} = x_t$, since each column must contain an instance of each variable. We may assume without loss of generality that this entry is positive (negate the row if necessary). This instance of x_t must have a nonconjugated preimage z_t. To see this, suppose to the contrary that $\psi(\mathbf{A})_{(s,j)} = x^*_t$. Since $\mathbf{A}_{(p,i)} = z_t$, the definition of Π from Lemma 4.3 implies $\mathbf{A}_{(p',u)} = \pm z^*_t$ for some $1 \leq u \leq 2m$. If $\psi(\mathbf{A})_{(s,j)} = x^*_t$, then the entries $\psi(\mathbf{A})_{(p',j)} = -x^*_q$ and $\psi(\mathbf{A})_{(p',u)} = \pm x^*_t$ would make orthogonality impossible in the preimage, a contradiction.

Now, if s' is the index of the row partner for row s of $\mathbf{A}$, then the SSSP applied specifically to the strong $\mathbf{B}_t$ submatrix implies that $\psi(\mathbf{A})_{(s',i)} = x^*_q$. So $\psi(\mathbf{A})_{(s,j)} = x_t$ and $\psi(\mathbf{A})_{(s',i)} = x^*_q$, which implies that $\Psi(\mathbf{C})_{(s',j)} = x_{t+k}$ and $\Psi(\mathbf{C})_{(s,i)} = -x^*_{q+k}$. Hence, the rows with indices s and s' provide a total contribution of $-x^*_{q+k}x_t + x^*_q x_{t+k}$ in $\mathbf{D}_{(i,j)}$. It follows that $\mathbf{D} = \mathbf{0}$, as required.

We have shown that the existence of a $(2k, 2m, k)$ BCOD implies the existence of a $(2k, 2m, 2k)$ ROD. Now, the tight lower bound of $\nu(n)$ on the decoding delay for rate 1 RODs [9], [28], [29] implies that $\nu(n)$ is also a lower bound on the decoding delay of BCODs. ■

Corollary 4.5: Let $\mathbf{A}$ be a BCOD. Then the lower bound of $\nu(n)$ on decoding delay of $\mathbf{A}$ is achievable when n is congruent to 2, 4, or 6 modulo 8; thus, $\nu(n)$ is the exact minimum decoding delay for BCODs with n columns when n is congruent to 2, 4, or 6 modulo 8 columns.

Proof: This follows directly from the modified-Liang algorithm of Section III and the fact that $\nu(n) = 2^m$ for $n = 2m$ congruent to 2, 4 and 6 modulo 8. ■

We reiterate that the modified-Liang algorithm also shows that the upper bound of $\nu(n)$ is achievable when the number of columns n is congruent to 3 or 5 modulo 8, though it does not demonstrate achievement when the number of columns is congruent to 0, 1, or 7 modulo 8; it still provides a delay of 2^m in these cases, however this is equal to $2\nu(n)$ for such values of n.

We conjecture that in the case of 0 modulo 8 columns, it is impossible to achieve this bound with *balanced* CODs. For BCODs, we conjecture that this case of 0 modulo 8 columns has a tight lower bound of $2\nu(n)$; in order to achieve $\nu(n)$ in this case, we must sacrifice, for example, conjugation-separation or introduce linear processing or irrational scalar coefficients.

Our major contribution in this section was to prove that $\nu(n)$ is the exact minimum decoding delay for a large and important class of rate 1/2 CODs. This leads us to conjecture that $\nu(n)$ is also the exact decoding delay of more general rate 1/2 CODs. This conjecture has been made independently by others, as well [10].

V. Low PAPR, Low Delay, Rate 1/2 CODs

In this section, we prove the existence of a class of rate 1/2 CODs that perform well with respect to design considerations C4–C7 and with respect to the guidelines of Yuen *et al.* [11]: they have no zero entries, each variable appearing exactly twice per column, effectively no irrational coefficients, and no LP. They also meet the conjectured lower bound on decoding delay in most cases. A distinguishing feature of these codes is that they are rich with algebraic and combinatorial structures. Furthermore, our construction technique is simple, relying only on the CODs obtained by our straight-forward modified-Liang algorithm in Section III. The technique does not require iterative building.

In the following Lemma 5.1, we will prove that any $(2k, 2m, k)$ BCOD $\mathbf{A}$ has a zero-masking row companion matrix $\mathbf{B}$ with some additional properties that will be required in the proof of Theorem 5.2, our main result in this section.

Lemma 5.1: Any $(2k, 2m, k)$ BCOD $\mathbf{A}$ has a $(2k, 2m, k)$ zero-masking row companion matrix $\mathbf{B}$ such that if row i of $\mathbf{A}$ is conjugated, then row i of $\mathbf{B}$ will be nonconjugated, and *vice versa*, for all $1 \leq i \leq 2k$.

Proof: This lemma follows directly from Lemma 4.3, however the following proof avoids reliance on the skew-symmetric matrix property in the definition of BCODs, giving a potentially more general result. It also provides more intuition regarding the structure of these designs, and it introduces some algebraic machinery that may be useful in future research regarding CODs.

We first define an equivalence relation on the rows of a BCOD. Suppose $\mathbf{r}$ and $\mathbf{s}$ are rows of a BCOD $\mathbf{A}$: define $\mathbf{r} \sim_{\mathbf{A}} \mathbf{s}$ if these rows have zeros in the same positions. We can easily see that $\sim_{\mathbf{A}}$ is an equivalence relation with equivalence classes $[\mathbf{r}]_{\mathbf{A}} = \{\mathbf{s} \text{ a row of } \mathbf{A} | \mathbf{s} \sim_{\mathbf{A}} \mathbf{r}\}$. We note that $\mathbf{r} \sim_{\mathbf{A}} \mathbf{s}$ if and only if $\phi(\mathbf{r}) \sim_{\phi(\mathbf{A})} \phi(\mathbf{s})$ for ϕ any combination of equivalence operations on the matrix $\mathbf{A}$. Similarly, rows $\mathbf{r}$ and $\mathbf{t}$ have *complementary* zero patterns if and only if $\phi(\mathbf{r})$ and $\phi(\mathbf{t})$ have complementary zero patterns. We can choose ϕ to represent the necessary operations to transform $\mathbf{A}$ into strong $\mathbf{B}_i$ form, make observations about what happens there, and transfer those observations back to the original matrix $\mathbf{A}$. For example, if $\mathbf{A}$ has $2m$ columns, then if some instance of z_i (or z_i^*) occupies a position in a row $\mathbf{r}$ in $\mathbf{A}$, then there is a row $\mathbf{t}$ in $\mathbf{A}$ that has complementary zero pattern to $\mathbf{r}$ that has an instance of z_i^* (or z_i) occupying one of its positions because that is true in strong $\mathbf{B}_i$ form. Thus, whenever $\mathbf{A}$ has $2m$ columns, every equivalence class $[\mathbf{r}]_{\mathbf{A}}$ will have a complementary equivalence class $[\mathbf{t}]_{\mathbf{A}}$ with the additional property that variables in $[\mathbf{t}]_{\mathbf{A}}$ correspond to conjugations of the variables in $[\mathbf{r}]_{\mathbf{A}}$ (up to sign). If some instance of z_i occupies a position in a row $\mathbf{r}$ and $\mathbf{s} \sim_{\mathbf{A}} \mathbf{r}$, then no instance of z_i can occupy a position in $\mathbf{s}$ (if it did, then when we would have two rows in strong $\mathbf{B}_i$ form with the same zero pattern, an impossibility). Thus, there are no repeated variables in any of the rows in an equivalence class. We will now use these observations to create a zero-masking companion matrix for the given BCOD $\mathbf{A}$.

The zero-masking row companion matrix $\mathbf{B}$ for the given BCOD $\mathbf{A}$ will be found by swapping the placement of certain pairs of rows of $\mathbf{A}$, where the pairs of rows of $\mathbf{A}$ will have complementary zero patterns and opposite conjugation. Given an equivalence class $[\mathbf{r}]_{\mathbf{A}}$, condition 3) of the definition of BCODs implies that any rows in this equivalence class contain variables that are either all conjugated or all nonconjugated. Given $\mathbf{r} \in [\mathbf{r}]_{\mathbf{A}}$, consider a row $\mathbf{t}$ with the complementary zero patterns; such a row $\mathbf{t}$ exists according to our discussion in the preceding paragraph. It also follows from the discussion above that for every nonconjugated variable z_i that appears in one of the rows of $[\mathbf{r}]_{\mathbf{A}}$, there is a conjugated z_i^* appearing in a row of $[\mathbf{t}]_{\mathbf{A}}$. Since there are no repeated variables within an equivalence class, this implies that the number of nonconjugated rows in $[\mathbf{r}]_{\mathbf{A}}$ is the same as the number of conjugated rows in $[\mathbf{t}]_{\mathbf{A}}$. We can pair them however we like. An analogous argument enables us to pair conjugated rows from $[\mathbf{r}]_{\mathbf{A}}$ with nonconjugated rows in $[\mathbf{t}]_{\mathbf{A}}$. Thus, there is a well-defined pairing of rows of $\mathbf{A}$ such that the rows within each pair have complementary zero patterns and opposite conjugation. Define the order 2 permutation π to swap rows within these pairs, and then use π to define the zero-masking row companion $\mathbf{B}$. In conclusion, we have shown that any $(2k, 2m, k)$ BCOD $\mathbf{A}$ has a $(2k, 2m, k)$ zero-masking row companion matrix $\mathbf{B}$. Furthermore, if row i of $\mathbf{A}$ is conjugated, then row i of $\mathbf{B}$ will be nonconjugated, and *vice versa*, for all $1 \leq i \leq 2k$. ■

We note for clarity that although the permutations π and Π guaranteed to exist by Lemmas 5.1 and 4.3, respectively, are very similar and can be identical in certain examples, the permutation Π represents a more specific pairing of rows that results in paired rows having the same variables; the existence of Π relies on the SSSP. We have observed, however, that all of our examples of conjugation-separated rate 1/2 CODs that have a zero-masking companion in which partner rows have opposite conjugation (thereby satisfying the requirements in the proof of Lemma 5.1), also have the SSSP.

Theorem 5.2: Let $\mathbf{A}$ be a $(2k, 2m, k)$ BCOD, and let $\mathbf{B}$ be a zero-masking row companion obtained via a permutation π as described in Lemma 5.1. Then, $\mathbf{A} + \sigma_{\mathbf{B}}(\mathbf{B})$ is a rate 1/2 $(2k, 2m, k)$ COD of type $(2, 2, \ldots, 2)$.

Proof: The following proof does not rely on the SSSP. Let $\mathbf{A}$ be a $(2k, 2m, k)$ BCOD on variables $z_1, \ldots, z_k$. Form a $(2k, 2m, k)$ zero-masking row companion matrix $\mathbf{B}$ on variables $z_1, \ldots, z_k$ using the order 2 permutation π that is guaranteed to exist by Lemma 5.1. Then $\mathbf{A}$ and $\mathbf{B}$ are valid BCODs with opposite zero patterns, and their rows have opposite conjugation. Also, since $\mathbf{B}$ is conjugation-separated, we can consider the BCOD $\sigma_{\mathbf{B}}(\mathbf{B})$.

Consider $\mathbf{A} + \sigma_{\mathbf{B}}(\mathbf{B})$. Note that due to the opposite zero patterns of $\mathbf{A}$ and $\mathbf{B}$, $\mathbf{A} + \sigma_{\mathbf{B}}(\mathbf{B})$ has exactly one complex variable in each position; there are no zero entries or linear combinations of complex variables. We will now show that $\mathbf{A} + \sigma_{\mathbf{B}}(\mathbf{B})$ is a $(2k, 2m, k)$ COD of type $(2, 2, \ldots, 2)$ on variables $z_1, \ldots, z_k$

$$\begin{aligned}
(\mathbf{A} + \sigma_{\mathbf{B}}(\mathbf{B}))^H(\mathbf{A} + \sigma_{\mathbf{B}}(\mathbf{B}))
&= \mathbf{A}^H\mathbf{A} + \sigma_{\mathbf{B}}(\mathbf{B})^H\sigma_{\mathbf{B}}(\mathbf{B}) \\
&\quad + \mathbf{A}^H\sigma_{\mathbf{B}}(\mathbf{B}) + \sigma_{\mathbf{B}}(\mathbf{B})^H\mathbf{A} \\
&= \sum_{l=1}^{k} |z_l|^2\mathbf{I}_{2m} + \sum_{l=1}^{k} |z_l|^2\mathbf{I}_{2m} \\
&\quad + \mathbf{A}^H\sigma_{\mathbf{B}}(\mathbf{B}) + \sigma_{\mathbf{B}}(\mathbf{B})^H\mathbf{A} \\
&= \sum_{l=1}^{k} 2|z_l|^2\mathbf{I}_{2m} \\
&\quad + \mathbf{A}^H\sigma_{\mathbf{B}}(\mathbf{B}) + \sigma_{\mathbf{B}}(\mathbf{B})^H\mathbf{A}.
\end{aligned}$$

It remains to show that $\mathbf{A}^H\sigma_{\mathbf{B}}(\mathbf{B}) + \sigma_{\mathbf{B}}(\mathbf{B})^H\mathbf{A} = \mathbf{0}$, or equivalently, that $\mathbf{A}^H\sigma_{\mathbf{B}}(\mathbf{B}) = -\sigma_{\mathbf{B}}(\mathbf{B})^H\mathbf{A}$. We will do this by showing that the (i, j) entry of $\mathbf{A}^H\sigma_{\mathbf{B}}(\mathbf{B})$ is equal to the negative of the (i, j) entry of $\sigma_{\mathbf{B}}(\mathbf{B})^H\mathbf{A}$.

The (i, j) entry of $\mathbf{A}^H\sigma_{\mathbf{B}}(\mathbf{B})$ is obtained as the dot product of the conjugation of the ith column of $\mathbf{A}$ and the jth column of $\sigma_{\mathbf{B}}(\mathbf{B})$. Let $\mathbf{A}_i$ represent the ith column of $\mathbf{A}$ for all $i = 1, 2, \ldots, 2m$, and then $\mathbf{A}_i^*$ represents the column obtained by conjugating all nonzero entries of $\mathbf{A}_i$. Then, the jth column of $\sigma_{\mathbf{B}}(\mathbf{B})$ can be expressed as $\sigma_{\mathbf{B}}(\pi(\mathbf{A}_j))$, as it is obtained by applying to $\mathbf{A}_j$ the order 2 permutation π that is guaranteed to exist by Lemma 5.1, and then by applying the sign function $\sigma_{\mathbf{B}}$ to this resulting column. Hence, the (i, j) entry of $\mathbf{A}^H\sigma_{\mathbf{B}}(\mathbf{B})$ is expressed as $\mathbf{A}_i^* \cdot \sigma_{\mathbf{B}}(\pi(\mathbf{A}_j))$. Similarly, the (i, j) entry of $\sigma_{\mathbf{B}}(\mathbf{B})^H\mathbf{A}$ is denoted by $\sigma_{\mathbf{B}}(\pi(\mathbf{A}_i))^* \cdot \mathbf{A}_j$.

Hence, we want to show that $\mathbf{A}_i^* \cdot \sigma_{\mathbf{B}}(\pi(\mathbf{A}_j)) = -\sigma_{\mathbf{B}}(\pi(\mathbf{A}_i))^* \cdot \mathbf{A}_j$.

We have

$$\begin{aligned}\mathbf{A}_i^* \cdot \sigma_{\mathbf{B}}(\pi(\mathbf{A}_j)) &= \pi(\mathbf{A}_i^*) \cdot \pi(\sigma_{\mathbf{B}}(\pi(\mathbf{A}_j))) & (1)\\ &= \pi(\mathbf{A}_i^*) \cdot \sigma_{\mathbf{A}}(\mathbf{A}_j) & (2)\\ &= \sigma_{\mathbf{B}}(\pi(\mathbf{A}_i^*)) \cdot \sigma_{\mathbf{B}}(\sigma_{\mathbf{A}}(\mathbf{A}_j)) & (3)\\ &= \sigma_{\mathbf{B}}(\pi(\mathbf{A}_i^*)) \cdot (-\mathbf{A}_j) & (4)\\ &= -\sigma_{\mathbf{B}}(\pi(\mathbf{A}_i))^* \cdot \mathbf{A}_j. & (5)\end{aligned}$$

Line (1) follows from the fact that the dot product of permuted vectors equals the dot product of the original vectors. To see (2), we observe that $\sigma_{\mathbf{A}}(\mathbf{A}_j) = \pi(\sigma_{\mathbf{B}}(\pi(\mathbf{A}_j)))$. Applying $\sigma_{\mathbf{B}}$ to both vectors in (2) does not change the dot product, showing (3). Finally, $\sigma_{\mathbf{B}}(\sigma_{\mathbf{A}}(\mathbf{A}_j)) = -\mathbf{A}_j$ implies (4). Now (5) follows directly.

Hence, $(\mathbf{A}+\sigma_{\mathbf{B}}(\mathbf{B}))^H(\mathbf{A}+\sigma_{\mathbf{B}}(\mathbf{B})) = \sum_{l=1}^{k} 2|z_l|^2\mathbf{I}_{2m}$, so $\mathbf{A}+\sigma_{\mathbf{B}}(\mathbf{B})$ is a rate 1/2 COD of type $(2,2,\ldots,2)$ on complex variables $z_1,\ldots,z_k$. Every variable appears exactly twice per column and there are no zero entries. ■

This theorem provides a construction algorithm which, when followed by column deletion as necessary, provides an important class of rate 1/2 CODs for any number of columns. We have shown in Section III that the BCOD building blocks exist and are readily obtainable for any number of columns n. (Though convenient, it is not required to use BCODs as the building blocks, but rather any rate 1/2 CODs such that appropriate zero masking companions exist.) Furthermore, for n congruent to 2, 3, 4, 5, or 6 modulo 8, these low PAPR codes satisfy the bound of $\nu(n)$ on decoding delay.

The resulting low PAPR codes are expected to be among the most useful rate 1/2 CODs due to their simple algorithm and their properties related to design considerations C2 and C4–C7. As an example of the construction algorithm, we consider the following BCOD $\mathbf{A}$ obtained via the modified-Liang algorithm

$$\mathbf{A} = \begin{pmatrix} z_1 & 0 & 0 & 0 & z_2 & z_3\\ 0 & z_1 & 0 & -z_2 & 0 & z_4\\ 0 & 0 & z_1 & -z_3 & -z_4 & 0\\ 0 & z_2^* & z_3^* & z_1^* & 0 & 0\\ -z_2^* & 0 & z_4^* & 0 & z_1^* & 0\\ -z_3^* & -z_4^* & 0 & 0 & 0 & z_1^*\\ z_4 & -z_3 & z_2 & 0 & 0 & 0\\ 0 & 0 & 0 & z_4^* & -z_3^* & z_2^* \end{pmatrix}.$$

Then, using the permutation $\pi\,(=\Pi) = (1,4)(2,5)(3,6)(7,8)$, we obtain a zero-masking companion $\mathbf{B}$. We obtain a desirable COD of type $(2,2,2,2)$ through the $\mathbf{A}+\sigma_{\mathbf{B}}(\mathbf{B})$ construction

$$\mathbf{A}+\sigma_{\mathbf{B}}(\mathbf{B}) = \begin{pmatrix} z_1 & -z_2^* & -z_3^* & -z_1^* & z_2 & z_3\\ z_2^* & z_1 & -z_4^* & -z_2 & -z_1^* & z_4\\ z_3^* & z_4^* & z_1 & -z_3 & -z_4 & -z_1^*\\ z_1 & z_2^* & z_3^* & z_1^* & z_2 & z_3\\ -z_2^* & z_1 & z_4^* & -z_2 & z_1^* & z_4\\ -z_3^* & -z_4^* & z_1 & -z_3 & -z_4 & z_1^*\\ z_4 & -z_3 & z_2 & -z_4^* & z_3^* & -z_2^*\\ z_4 & -z_3 & z_2 & z_4^* & -z_3^* & z_2^* \end{pmatrix}.$$

VI. Conclusion

We have shown that the upper bound of $\nu(n)$ on the decoding delay of rate 1/2 CODs is also the lower bound on decoding delay for rate 1/2 balanced complex orthogonal designs (BCODs). Furthermore, it is the exact minimum decoding delay for BCODs with $n \neq 8k$ columns. BCODs achieve transceiver linearization, are power-balanced, have no irrational coefficients, no linear processing, and an even number of columns. Due to these properties, we expect that they will be among the most suitable for applications, as well as being among the most interesting combinatorially.

We also presented a family of type $(2,\ldots,2)$ rate 1/2 CODs that have low PAPR due to no zero entries, are power-balanced with each variable appearing exactly twice per column, effectively no irrational coefficients, and no linear processing. They achieve a decoding delay of $\nu(n)$ for most equivalence classes of n modulo 8. These CODs are obtained via an elegant algorithm that uses the algebraic and combinatorial properties of designs, rather than using iterative algorithms as done previously.

Although we presented achieving a decoding delay of $\nu(n)$ as a positive quality for rate 1/2 CODs, it is important to stress that the bound of $\nu(n)$ grows exponentially with the number of columns. While this is a significant improvement over the factorial growth of the delay of maximum rate CODs with only a small sacrifice in rate, it is still too large for practical applications with large numbers of antennas.

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Sarah Spence Adams (M’03) received the B.S. degree in mathematics (*summa cum laude*) from the University of Richmond, VA, and the M.S. and Ph.D. degrees in mathematics from Cornell University, Ithaca, NY, where she was also a member of the Wireless Intelligent Systems Laboratory in the Department of Electrical and Computer Engineering.

Previous nonacademic experience includes appointments at the Institute for Defense Analyses, Center for Communications Research and the National Security Agency. She is currently an Associate Professor of Mathematics and Electrical and Computer Engineering at Franklin W. Olin College of Engineering, Needham, MA. Research interests include discrete mathematics, algebraic error-control codes, space-time block codes, and other orthogonal matrices over finite and infinite alphabets.

James A. Davis received the B.S. degree in mathematics from Lafayette College, Easton, PA, and the M.S. and Ph.D. degrees in mathematics from the University of Virginia, Charlottesville.

He has been at the University of Richmond, VA, since 1988. His research interests include difference sets and their generalizations and applications.

Nathaniel Karst received the B.S. degree in electrical and computer engineering from Franklin W. Olin College of Engineering, Needham, MA. He is currently working toward the M.S. degree at the Center for Applied Mathematics, Cornell University, Ithaca, NY.

Mathav Kishore Murugan received the B.Tech. degree in electronics and electrical communication engineering and the M.Tech. degree in microelectronics and VLSI design from the Indian Institute of Technology, Kharagpur. He is currently working toward the Ph.D. degree at the Center for Applied Mathematics, Cornell University, Ithaca, NY.

Bryce Lee received the B.S. degree in mechanical engineering from Franklin W. Olin College of Engineering, Needham, MA. He is currently working toward the M.S. degree in the Mechanical Engineering Department, Virginia Tech, Blacksburg.

Matthew Crawford received the B.S. degree in electrical and computer engineering from Franklin W. Olin College of Engineering, Needham, MA.

He is currently with Athenahealth, Inc., Watertown, MA.

Caitlin Greeley received the B.S. degree in engineering with a concentration in material science from Franklin W. Olin College of Engineering, Needham, MA.

She worked in Market Risk at Goldman Sachs and is currently working in the Retail and Business Banking Division at Oliver Wyman, New York.