The Final Case of the Decoding Delay Problem for Maximum Rate Complex Orthogonal Designs

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***Abstract*—Complex orthogonal space–time block codes (COSTBCs) based on generalized complex orthogonal designs (CODs) have been successfully implemented in wireless systems with multiple transmit antennas and single or multiple receive antennas. It has been shown that for a maximum rate COD with $2m-1$ or $2m$ columns, a lower bound on decoding delay is $\binom{2m}{m-1}$ and this delay is achievable when the number of columns is congruent to 0, 1, or 3 modulo 4. In this paper, the final case is addressed, and it is shown that when the number of columns is congruent to 2 modulo 4, the lower bound on decoding delay cannot be achieved. In this case, the shortest decoding delay a maximum rate COD can achieve is twice the lower bound. New techniques for analyzing CODs are introduced with connections to binary vector spaces.**

***Index Terms*—Complex orthogonal designs, decoding delay, diversity, multiple-input multiple-output (MIMO) systems, space–time block codes.**

I. Introduction

COMPLEX orthogonal space–time block codes (COSTBCs) based on generalized complex orthogonal designs (CODs) are attractive because they permit a simple maximum-likelihood decoding rule and guarantee full diversity [1]. Complex orthogonal designs have been defined in a variety of ways [2]–[6], and the following generalization, which has proven useful in signals processing, will be used in this paper: An $[r, n, k]$ COD is an $r \times n$ matrix $\mathbf{G}$ with entries from $\{0, \pm z_1, \ldots, \pm z_k, \pm z_1^*, \ldots, \pm z_k^*\}$ such that $\mathbf{G}^H\mathbf{G} = \sum_{l=1}^{k} |z_l|^2 \mathbf{I}_n$, where H is the Hermitian transpose and $\mathbf{I}_n$ is the $n \times n$ identity matrix [1], [7]. In this definition of CODs, the designs are said to be *combinatorial*, in the sense that there is no linear processing permitted in the entries. Each variable z_i, $1 \le i \le k$, appears exactly once per column and at most once per row. When applied as a COSTBC, each column contains the transmission data for a distinct antenna, and each row contains the transmission data for a distinct timestep. Geramita and Seberry provide a comprehensive review of classical orthogonal designs [8], and Liang reviews and defines their generalizations [9].

Of key interest when studying COSTBCs, or equivalently their underlying CODs, are the *rate*, defined as the ratio $R = k/r$ of the number of information symbols (i.e., variables) to the decoding delay (i.e., number of rows), and the minimum decoding delay (i.e., minimum number of rows) achievable for a given rate.

Liang determined that the maximum rate for a COD with $2m-1$ or $2m$ columns is $R_0 = \frac{m+1}{2m}$, where m is any natural number [7]. Furthermore, Liang provided an algorithm for constructing maximum rate CODs for any number of columns [7]. Several other authors have also worked towards determining the maximum rate and developing algorithms to produce high-rate CODs [10]–[12].

Adams, Karst, and Pollack showed that a lower bound on decoding delay for maximum rate CODs with $2m-1$ or $2m$ columns is $r_0 = \binom{2m}{m-1}$ [13]. The algorithm by Lu, Fu, and Xia proves that this bound is achievable if the number of columns is congruent to 0, 1, or 3 modulo 4 [14]. Previously, only special cases had been determined, generally using exhaustive techniques specialized to the specific number of columns involved, but conjectures had been made concerning the general result. The cases with fewer than 5 columns are trivial. Liang addressed CODs with 5 and 6 columns, and he left as an open problem whether the formula $\frac{2m}{m+1}\binom{n}{m}$ held for $n = 2m-1$ and $n = 2m$ for $n > 6$ as he showed it held for $n \le 6$, $n \ne 4$ [7]. Kan and Shen addressed the lower bound on delay for CODs with seven and eight columns [15], and Liang later confirmed the eight-column case using a padding argument [16]. Kan and Shen's work showed that Liang's conjecture needed to be modified. They conjectured to change the case when 4 divides n to $\frac{m}{m+1}\binom{n}{m}$ [15]. This conjecture by Kan and Shen is numerically equivalent to the conjecture in [13], and it agrees with the results obtained therein for the cases of n congruent to 0, 1, and 3 modulo 4 [13]; these works developed independent of each other and were focused on proving the lower bound for decoding delay. The upper bound, through the development of algorithms, has already been settled [7], [14].

The work in this paper closes the final case concerning the minimum achievable decoding delay when the number of columns is congruent to 2 modulo 4. The result is consistent with our previous conjecture [13] and the conjecture by Kan and Shen [15], which is a modification of the original conjecture by Liang [7].

The case where the number of columns is congruent to 2 modulo 4 requires novel proof techniques, though it does build on our previous work. Prior results show that if a maximum rate

Manuscript received March 28, 2008; revised January 23, 2009. Current version published December 23, 2009. The work of S. S. Adams was supported in part by the NSA by Grant H98230-07-1-0022. The work of M. K. Murugan was supported in part by Olin College, Needham, MA.

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Communicated by B. S. Rajan, Associate Editor for Coding Theory.

Digital Object Identifier 10.1109/TIT.2009.2034818

COD does not meet the lower bound on decoding delay, then its delay must be an integer multiple of the lower bound [13]. Indeed, the best algorithms to date for maximum rate CODs with $n \equiv 2$ modulo 4 columns produce examples that achieve twice the lower bound on delay [7], [12], [14]. In this paper, we close this final case by proving that the best achievable delay for maximum rate designs with $2m$ columns, m odd, is $2r_0 = 2\binom{2m}{m-1}$, or twice the lower bound. We conclude that for $n \not\equiv 2$ modulo 4, a maximum rate, minimum decoding delay COD has parameters $[r_0, n, r_0R_0]$, and for $n \equiv 2$ modulo 4, a maximum rate, minimum decoding delay COD has parameters $[2r_0, n, 2r_0R_0]$.

Section II outlines some preliminary notation that is used in the balance of the paper. In Section III, we introduce a *standard form* for $[r_0, 2m-1, r_0R_0]$ CODs that determines the sign of each entry within the maximum rate, minimum decoding delay COD. Section IV uses this standard form to prove that maximum rate CODs with $2m$ columns for m odd cannot achieve the lower bound on decoding delay. It is then implied that the best achievable delay in this case is twice the lower bound. The paper is concluded in Section V.

II. Preliminary Notation

We say that an entry of a COD contains an *instance* of a variable z if the entry is from $\{\pm z, \pm z^*\}$. Liang [7] noted that given a maximum rate COD $\mathbf{G}$ with $2m-1$ or $2m$ columns over variables $z_1, \ldots, z_k$, for any z_i, $1 \le i \le k$, there exist suitable *equivalence operations* (i.e., rearrangements of rows and columns, multiplications of rows and columns by -1, and conjugation and/or negation of all instances of certain variables) to transform $\mathbf{G}$ such that the first $2m-1$ or $2m$ (respectively) rows of $\mathbf{G}$ are of the form:

$$\mathbf{B}_i = \begin{pmatrix} z_i\mathbf{I} & \mathbf{M}_i \\ -\mathbf{M}_i^H & z_i^*\mathbf{I} \end{pmatrix}.$$

Liang further showed that if $\mathbf{G}$ has $2m$ columns, then $\mathbf{M}_i$ is $m \times m$. If $\mathbf{G}$ has $2m-1$ columns, then $\mathbf{M}_i$ is either $(m-1) \times m$ or $m \times (m-1)$. In this case, we will assume that $\mathbf{M}_i$ is $m \times (m-1)$; all proofs can be altered slightly for the alternative. We will also use Liang's result that the $\mathbf{M}_i$ submatrices have no zero entries [7].

Let any (possibly noncontiguous) 2×2 orthogonal submatrix of a COD that is isomorphic under equivalence operations to Alamouti's original COD

$$\mathbf{A} = \begin{pmatrix} z_i & z_j \\ -z_j^* & z_i^* \end{pmatrix}$$

be called an *Alamouti* 2×2. Throughout this paper, we will use the fact that any Alamouti 2×2 must contain an odd number of negative entries and one conjugated and one nonconjugated appearance of each of the two included variables.

Let any (possibly noncontiguous) 2×2 orthogonal submatrix that is isomorphic under equivalence operations to the following matrix

$$\mathbf{T} = \begin{pmatrix} z_i & 0 \\ 0 & z_i \end{pmatrix}$$

be called a *trivial* 2×2. Since $\mathbf{M}_i$ has no zero entries, any trivial 2×2 must contain either two conjugated or two nonconjugated entries.

We say that two rows of a COD *share* an Alamouti or a trivial 2×2 over two columns if the intersection of these rows and columns forms such a 2×2 orthogonal submatrix.

III. Standard Form for $[r_0, 2m-1, r_0R_0]$ CODs

In this section, we present a standard form for maximum rate, minimum delay CODs with $2m-1$ columns, or in other words, for $[r_0, 2m-1, r_0R_0]$ CODs. The standard form dictates only the sign of each entry, and we will show in Theorem 3.7 that it is achieved through the equivalence operations of negating all instances of certain variables ("instance negations") and negating all entries in certain rows ("row negations"). If a COD $\mathbf{G}$ undergoes such equivalence operations in order to achieve standard form, we abuse notation when no confusion should occur and denote this arrangement of the COD again by $\mathbf{G}$. To define this standard form, we need the following notation.

Given an $[r, n, k]$ COD $\mathbf{G}$, for $1 \le i \le n$, let $\mathbf{c}_i$ denote the ith column of $\mathbf{G}$, and for $1 \le s \le r$, let $\mathbf{r}_s$ denote the sth row of $\mathbf{G}$. Define the *support* of $\mathbf{r}_s$ as a binary vector $\tilde{\mathbf{r}}_s$ of length n that contains a 1 in every position in which $\mathbf{r}_s$ is nonzero and a 0 in every position in which $\mathbf{r}_s$ is zero.

We now introduce a set of basis vectors that will be used to define the standard form. Define $\mathbf{v}_{i,j}, i \ne j$, to be a binary vector of length $2m-1$ that contains 0 in positions i and j, and 1 in its remaining $2m-3$ positions. Note that $\mathbf{v}_{i,j} = \mathbf{v}_{j,i}$. Define $\mathbf{v}_{i,i}$ to be a binary vector of length $2m-1$ that contains a 1 in position i and 0 in the remaining $2m-2$ positions. In the interest of space, the details are left to the reader to show that for each $1 \le i \le 2m-1$, the set of vectors $\mathbf{V}_i = \{\mathbf{v}_{i,j} \mid j = 1, \ldots, 2m-1\}$ forms a basis of the $(2m-1)$-dimensional binary vector space.

It now follows that any support $\tilde{\mathbf{r}}_s$ for a row $\mathbf{r}_s$ of length $2m-1$ can be written as a linear combination of vectors in $\mathbf{V}_i$, for any $1 \le i \le 2m-1$. For a given basis $\mathbf{V}_i$ and a given binary vector $\tilde{\mathbf{r}}_s$, we write $\tilde{\mathbf{r}}_s = a_1^{s,i}\mathbf{v}_{i,1} + a_2^{s,i}\mathbf{v}_{i,2} + \cdots + a_{2m-1}^{s,i}\mathbf{v}_{i,2m-1}$. We call this the vector expansion of $\tilde{\mathbf{r}}_s$ over $\mathbf{V}_i$.

Definition 3.1 presents the formal definition of the standard form of a $[r_0, 2m-1, r_0R_0]$ COD $\mathbf{G}$. Qualitatively, the standard form is defined so that for each Alamouti 2×2 submatrix $\mathbf{A}$ of $\mathbf{G}$, the entries in the left-hand column of $\mathbf{A}$ have the same sign and the entries in the right-hand column of $\mathbf{A}$ have different signs.

Definition 3.1: An $[r_0, 2m-1, r_0R_0]$ COD $\mathbf{G}$ is said to be in *standard form* if the signs of the nonzero entries of $\mathbf{G}$ are as follows. All nonzero entries in column $\mathbf{c}_1$ are positive. A nonzero entry $\mathbf{G}(s,i)$ at the intersection of row $\mathbf{r}_s$, $1 \le s \le r_0$, and column $\mathbf{c}_i$, $2 \le i \le 2m-1$, has its sign dictated by the vector expansion of $\tilde{\mathbf{r}}_s$ over $\mathbf{V}_i$. If the weight of the vector consisting of the first $i-1$ binary coefficients of this vector expansion, denoted $w\left(a_1^{s,i}, \ldots, a_{i-1}^{s,i}\right)$, is congruent to 0 modulo 2, then $\mathbf{G}(s,i)$ is positive; if $w\left(a_1^{s,i}, \ldots, a_{i-1}^{s,i}\right) \equiv 1$ modulo 2, then $\mathbf{G}(s,i)$ is negative.

If the sign(s) of an entry/column/matrix follow(s) the rules in Definition 3.1, we say that the entry/column/matrix is in standard form, or simply that the entry/column/matrix has the correct sign(s).

To show that any $[r_0, 2m-1, r_0R_0]$ COD can be put into standard form, we first must formalize in the following lemmas some connections among the definition of standard form, the bases $\mathbf{V}_i$, and the conditions under which two rows share an Alamouti or trivial 2×2.

Lemma 3.2: Let $\mathbf{r}_s$ and $\mathbf{r}_t$ be distinct rows of an $[r_0, n, r_0R_0]$ COD $\mathbf{G}$. Then, we have the following.

1) Rows $\mathbf{r}_s$ and $\mathbf{r}_t$ share an *Alamouti* 2×2 over columns $\mathbf{c}_h$ and $\mathbf{c}_i$ if and only if $\mathbf{r}_s$ and $\mathbf{r}_t$ are simultaneously nonzero exactly in columns $\mathbf{c}_h$ and $\mathbf{c}_i$ and never simultaneously zero in any column. If $n = 2m-1$ and $i > h$, then the standard form signs of entries $\mathbf{G}(s,i)$ and $\mathbf{G}(t,i)$ are different and the standard form signs of $\mathbf{G}(s,h)$ and $\mathbf{G}(t,h)$ are the same.
2) Rows $\mathbf{r}_s$ and $\mathbf{r}_t$ share a *trivial* 2×2 over columns $\mathbf{c}_h$ and $\mathbf{c}_i$ if and only if $\mathbf{r}_s$ and $\mathbf{r}_t$ are simultaneously zero or simultaneously nonzero in all columns except $\mathbf{c}_h$ and $\mathbf{c}_i$. If $n = 2m-1$ and $i > h$, then for i even, $w\left(a_1^{s,i}, \ldots, a_{i-1}^{s,i}\right) \equiv w\left(a_1^{t,i}, \ldots, a_{i-1}^{t,i}\right)$ modulo 2, and for i odd, $w(a_1^{s,i}, \ldots, a_{i-1}^{s,i}) \not\equiv w\left(a_1^{t,i}, \ldots, a_{i-1}^{t,i}\right)$ modulo 2.

Proof: See Appendix. □

The following definition and lemma further formalize some properties of rows that share Alamouti or trivial 2×2's.

Definition 3.3: Let $\mathbf{G}$ be an $[r_0, 2m-1, r_0R_0]$ COD. Recursively define a 2×2 set with respect to column $\mathbf{c}_i$ using seed row $\mathbf{r}$, denoted $S_{\mathbf{r}}(\mathbf{c}_i)$, to be a nonempty subset of rows of $\mathbf{G}$ obtained as follows: Choose an arbitrary row $\mathbf{r}$ in $\mathbf{G}$ as the initial seed for the set $S_{\mathbf{r}}(\mathbf{c}_i)$; then include in $S_{\mathbf{r}}(\mathbf{c}_i)$ all rows of $\mathbf{G}$ that share an Alamouti or trivial 2×2 with $\mathbf{r}$ over columns $\mathbf{c}_h$ and $\mathbf{c}_i$, for all $h < i$; then include all rows that share an Alamouti or trivial 2×2 with any of these rows over columns $\mathbf{c}_h$ and $\mathbf{c}_i$, for all $h < i$; continue this process until it does not produce any new rows, and include each generated row exactly once in the final set $S_{\mathbf{r}}(\mathbf{c}_i)$.

This recursive definition of a 2×2 set with respect to column $\mathbf{c}_i$ is well defined, so that a given 2×2 set can be generated using any of its rows as the initial seed. This follows immediately from the symmetry of the relationship of two rows sharing an Alamouti or trivial 2×2. When no confusion should occur (i.e., as the choice of seed row is irrelevant), we write simply $S(\mathbf{c}_i)$.

Example 3.4: The following matrix $\mathbf{G}_1$ is a $[4,3,3]$ COD whose rows have been partitioned with a horizontal line into 2×2 sets with respect to column $\mathbf{c}_2$:

$$\mathbf{G}_1 = \left(\begin{array}{ccc} z_1 & -z_2 & z_3 \\ z_2^* & z_1^* & 0 \\ \hline 0 & -z_3^* & -z_2^* \\ z_3^* & 0 & -z_1^* \end{array}\right).$$

The reader can confirm that $S_{\mathbf{r}_1}(\mathbf{c}_2) = S_{\mathbf{r}_2}(\mathbf{c}_2)$, and similarly $S_{\mathbf{r}_3}(\mathbf{c}_2) = S_{\mathbf{r}_4}(\mathbf{c}_2)$. We also note that for this COD $\mathbf{G}_1$, any 2×2 set with respect to column $\mathbf{c}_3$ includes all four rows of the COD.

Example 3.5: The following matrix $\mathbf{G}_2$ is a $[15,5,10]$ COD whose rows have been partitioned with horizontal lines into 2×2 sets with respect to column $\mathbf{c}_3$:

$$\mathbf{G}_2 = \left(\begin{array}{ccccc} z_2^* & -z_5^* & -z_1^* & 0 & 0 \\ 0 & -z_1 & z_5 & -z_6 & -z_7 \\ z_1 & 0 & z_2 & -z_3 & -z_4 \\ z_5 & z_2 & 0 & z_8 & z_9 \\ \hline z_6 & z_3 & z_8 & 0 & z_{10} \\ 0 & z_8^* & -z_3^* & -z_2^* & 0 \\ z_8^* & 0 & -z_6^* & -z_5^* & 0 \\ z_3^* & -z_6^* & 0 & z_1^* & 0 \\ \hline z_7 & z_4 & z_9 & -z_{10} & 0 \\ 0 & z_9^* & -z_4^* & 0 & -z_2^* \\ z_9^* & 0 & -z_7^* & 0 & -z_5^* \\ z_4^* & -z_7^* & 0 & 0 & z_1^* \\ \hline z_{10}^* & 0 & 0 & z_7^* & -z_6^* \\ 0 & z_{10}^* & 0 & z_4^* & -z_3^* \\ 0 & 0 & z_{10}^* & z_9^* & -z_8^* \end{array}\right).$$

Lemma 3.6: Consider an $[r_0, 2m-1, r_0R_0]$ COD $\mathbf{G}$. Then,

1) If for some $2 \le i \le 2m-1$, columns $\mathbf{c}_1, \ldots, \mathbf{c}_{i-1}$ are in standard form and if one entry in column $\mathbf{c}_i$ within a row of $S(\mathbf{c}_i)$ has the incorrect sign with respect to the standard form, then all entries in $\mathbf{c}_i$ within the rows of $S(\mathbf{c}_i)$ have the incorrect signs.
2) If for some $2 \le i \le 2m-1$, column $\mathbf{c}_i$ contains an instance of a variable z within a row $\mathbf{r}_t$ in $S(\mathbf{c}_i)$, then $S(\mathbf{c}_i)$ contains all instances of z found in columns $\mathbf{c}_1, \ldots, \mathbf{c}_{i-1}$, and these instances occur precisely in rows that share either an Alamouti or trivial 2×2 with $\mathbf{r}_t$ over column $\mathbf{c}_i$ and some column $\mathbf{c}_h$, $h < i$.
3) For any given $2 \le i \le 2m-1$, distinct 2×2 sets for column $\mathbf{c}_i$ are disjoint and the union of all 2×2 sets for $\mathbf{c}_i$ contains all rows in $\mathbf{G}$.

Proof: See Appendix. □

We now leverage the previous lemmas to show that any $[r_0, 2m-1, r_0R_0]$ COD can be put into standard form through the equivalence operations of instance negations and row negations.

Theorem 3.7: Let $\mathbf{G}$ be an $[r_0, 2m-1, r_0R_0]$ COD. There exists a sequence of equivalence operations to place $\mathbf{G}$ in standard form by sequentially placing each column $\mathbf{c}_1, \ldots, \mathbf{c}_{2m-1}$ in standard form.

Proof: We will prove this theorem using strong mathematical induction. For the base case, put $\mathbf{c}_1$ in standard form by negating all rows in which $\mathbf{c}_1$ originally had a negative entry. Now assume that columns $\mathbf{c}_1, \ldots, \mathbf{c}_{i-1}$ are in standard form for some $2 \le i \le 2m-1$. We will show that this implies $\mathbf{c}_1, \ldots, \mathbf{c}_i$ can be put into standard form through a specific series of equivalence operations.

Consider a 2×2 set $S(\mathbf{c}_i)$ with respect to column $\mathbf{c}_i$. By Part 1) of Lemma 3.6, we may assume that $S(\mathbf{c}_i)$ is a 2×2 set in which all the nonzero entries in column $\mathbf{c}_i$ within the rows of $S(\mathbf{c}_i)$ have incorrect signs. Then, given a row $\mathbf{r}_s$ in $S(\mathbf{c}_i)$ where $w\left(a_1^{s,i}, \ldots, a_{i-1}^{s,i}\right) \equiv 0$ modulo 2 and which contains an instance of some variable z in column $\mathbf{c}_i$, negate all instances of the variable z within $\mathbf{G}$. This instance negation corrects the sign on the entry z in $\mathbf{G}(s,i)$, but it makes the sign of any instance of z in columns $\mathbf{c}_1, \ldots, \mathbf{c}_{i-1}$ incorrect. Hence, all entries in columns $\mathbf{c}_1, \ldots, \mathbf{c}_i$ within row $\mathbf{r}_s$ are now correct, where $\mathbf{r}_s$ is a row in $S(\mathbf{c}_i)$ where $w\left(a_1^{s,i}, \ldots, a_{i-1}^{s,i}\right) \equiv 0$ modulo 2and where $\mathbf{G}(s,i)$ is nonzero. Repeat this procedure by negating all instances of any variable that appears in column $\mathbf{c}_i$ within any row $\mathbf{r}_t$ in $S(\mathbf{c}_i)$ where $w\left(a_1^{t,i}, \ldots, a_{i-1}^{t,i}\right) \equiv 0$ modulo 2. It follows from Part 2) of Lemma 3.6 and from the proof of Lemma 3.2 that repeating this procedure on such rows does not harm the standard form of the entries in columns $\mathbf{c}_1, \ldots, \mathbf{c}_{i-1}$ within any of the considered rows. So, at this point, correct signs have been restored/obtained on all entries in columns $\mathbf{c}_1, \ldots, \mathbf{c}_i$ within any row $\mathbf{r}_t$ such that $w\left(a_1^{t,i}, \ldots, a_{i-1}^{t,i}\right) \equiv 0$ modulo 2and $\mathbf{G}(t,i)$ is nonzero. This completes what we call the instance-negation phase of the algorithm.

In the second and final phase of the algorithm, dubbed the correction phase of the algorithm, we correct all entries in columns $\mathbf{c}_1, \ldots, \mathbf{c}_{i-1}$ that were made incorrect during the instance-negation phase of the algorithm. Specifically, by Part 2) of Lemma 3.6, the entries to be corrected are in rows $\mathbf{r}_u$ that share Alamouti or trivial 2×2's over columns $\mathbf{c}_i$ and some $\mathbf{c}_h$, $h < i$, with such above-described rows $\mathbf{r}_t$ of $S(\mathbf{c}_i)$.

For the first step of the correction phase of the algorithm, suppose that $\mathbf{r}_u$ is a row of $S(\mathbf{c}_i)$ that shares an Alamouti 2×2 with some such above-described row $\mathbf{r}_t$ over columns $\mathbf{c}_h$ and $\mathbf{c}_i$ for some $h < i$. Suppose further that entry $\mathbf{G}(t,i)$ contains an instance of a variable z. Then, by the definition of an Alamouti 2×2, entry $\mathbf{G}(u,h)$ contains an instance of the variable z. It then follows that the instance of z in entry $\mathbf{G}(u,h)$ was given the incorrect sign during the instance-negation phase of the algorithm. In fact, we will show that *all* nonzero entries in columns $\mathbf{c}_1, \ldots, \mathbf{c}_{i-1}$ within $\mathbf{r}_u$ have the incorrect sign.

For any $1 \leq k \leq i-1$ such that $\mathbf{G}(u,k)$ is nonzero, consider the variable in this entry, which we will call x. Since row $\mathbf{r}_u$ shares an Alamouti 2×2 with row $\mathbf{r}_t$ over columns $\mathbf{c}_h$ and $\mathbf{c}_i$, the entry $\mathbf{G}(u,i)$ is nonzero; say it contains an instance of some variable w. To maintain orthogonality of $\mathbf{G}$, there must be some other row $\mathbf{r}_v$ that shares an Alamouti 2×2 with row $\mathbf{r}_u$ over columns $\mathbf{c}_k$ and $\mathbf{c}_i$, such that an instance of w is in entry $\mathbf{G}(v,k)$ and an instance of x is in entry $\mathbf{G}(v,i)$. We have assumed that $w\left(a_1^{t,i}, \ldots, a_{i-1}^{t,i}\right) \equiv 0$ modulo 2, so since $i > h$, the proof of Lemma 3.2 indicates that $w\left(a_1^{u,i}, \ldots, a_{i-1}^{u,i}\right) \equiv 1$ modulo 2, and then since $i > k$, that $w\left(a_1^{v,i}, \ldots, a_{i-1}^{v,i}\right) \equiv 0$ modulo 2.

So, we have shown that $\mathbf{r}_v$ is a row in $S(\mathbf{c}_i)$ with $w\left(a_1^{v,i}, \ldots, a_{i-1}^{v,i}\right) \equiv 0$ modulo 2 and with a nonzero instance of x in column $\mathbf{c}_i$, hence all instances of x were negated during the instance-negation phase of this algorithm. So, the instance of x in entry $\mathbf{G}(u,k)$, $k < i$, currently has the incorrect sign. This implies that all nonzero entries $\mathbf{G}(u,k)$ for $1 \leq k \leq i-1$ have the incorrect sign. To see that $\mathbf{G}(u,i)$ also has the incorrect sign, note that $\mathbf{r}_u$ is a row in $S(\mathbf{c}_i)$ with $w\left(a_1^{u,i}, \ldots, a_{i-1}^{u,i}\right) \equiv 1$ modulo 2, so the instance of w in $\mathbf{G}(u,i)$ was *not* negated during the negation-phase of the algorithm; hence, this instance of w has the incorrect sign due to our initial assumption that all nonzero entries in column $\mathbf{c}_i$ within rows of $S(\mathbf{c}_i)$ have the incorrect sign. Thus, we have shown that the signs are incorrect for all nonzero entries in columns $\mathbf{c}_1, \ldots, \mathbf{c}_i$ within any row $\mathbf{r}_u$ in $S(\mathbf{c}_i)$ that shares an Alamouti 2×2 with any row $\mathbf{r}_t$ of $S(\mathbf{c}_i)$ that has $w\left(a_1^{t,i}, \ldots, a_{i-1}^{t,i}\right) \equiv 0$ modulo 2 and $\mathbf{G}(t,i)$ is nonzero.

Hence, to restore/obtain correct signs in nonzero entries located in columns $\mathbf{c}_1, \ldots, \mathbf{c}_i$ within any row $\mathbf{r}_u$ in $S(\mathbf{c}_i)$ that shares an Alamouti 2×2 with $\mathbf{r}_t$ (where $\mathbf{r}_t$ is in $S(\mathbf{c}_i)$, $w\left(a_1^{t,i}, \ldots, a_{i-1}^{t,i}\right) \equiv 0$ modulo 2, and $\mathbf{G}(t,i)$ is nonzero), simply negate row $\mathbf{r}_u$. Equivalently, negate all rows $\mathbf{r}_u$ in $S(\mathbf{c}_i)$ such that $w\left(a_1^{u,i}, \ldots, a_{i-1}^{u,i}\right) \equiv 1$ modulo 2 and $\mathbf{G}(u,i)$ is nonzero.

Now, for the second and final step of the correction phase of the algorithm, suppose that $\mathbf{r}_u$ is a row of $S(\mathbf{c}_i)$ that shares a trivial 2×2 with $\mathbf{r}_t$ (where $\mathbf{r}_t$ is as described above) over columns $\mathbf{c}_h$ and $\mathbf{c}_i$ for some $h < i$. Then, by the definition of trivial 2×2's, and since $\mathbf{G}(t,i)$ contains an instance of z, $\mathbf{G}(u,i)$ must be zero and $\mathbf{G}(u,h)$ must contain an instance of z. Since columns $\mathbf{c}_1, \ldots, \mathbf{c}_{i-1}$ were assumed to be in standard form before the instance-negation phase of this algorithm, and since all instances of z were negated during that phase, the instance of z found in entry $\mathbf{G}(u,h)$ currently has the incorrect sign. Similar to the case of shared Alamouti 2×2's, we will show that *all* nonzero entries in columns $\mathbf{c}_1, \ldots, \mathbf{c}_{i-1}$ within $\mathbf{r}_u$ have the incorrect sign: For any $1 \leq k \leq i-1$ such that $\mathbf{G}(u,k)$ is nonzero, we again consider the variable in this entry and call it x. The variable x must appear somewhere in column $\mathbf{c}_i$, say in row $\mathbf{r}_v$. Then, since $\mathbf{G}(u,i) = 0$, the orthogonality of $\mathbf{G}$ implies that rows $\mathbf{r}_u$ and $\mathbf{r}_v$ share a trivial 2×2 over columns $\mathbf{c}_k$ and $\mathbf{c}_i$. Since we assumed $w\left(a_1^{t,i}, \ldots, a_{i-1}^{t,i}\right) \equiv 0$ modulo 2, Lemma 3.2 shows that if i is odd, then since $i > h$, $w\left(a_1^{u,i}, \ldots, a_{i-1}^{u,i}\right) \equiv 1$ modulo 2, and then since $i > k$, that $w\left(a_1^{v,i}, \ldots, a_{i-1}^{v,i}\right) \equiv 0$ modulo 2. Similarly, if i is even, Lemma 3.2 implies that $w\left(a_1^{u,i}, \ldots, a_{i-1}^{u,i}\right) \equiv 0$ modulo 2 and then again that $w\left(a_1^{v,i}, \ldots, a_{i-1}^{v,i}\right) \equiv 0$ modulo 2. So, regardless of parity of i, we have $w\left(a_1^{v,i}, \ldots, a_{i-1}^{v,i}\right) \equiv 0$ modulo 2. This implies that since $\mathbf{G}(v,i)$ contains an instance of x, all instances of the variable x were negated in the instance-negation phase of this algorithm. This implies that the instance of x in entry $\mathbf{G}(u,k)$ currently has the incorrect sign. (The row $\mathbf{r}_u$, hence the entry $\mathbf{G}(u,k)$, was not corrected in any step above, as all rows corrected above had nonzero entries in column $\mathbf{c}_i$ while row $\mathbf{r}_u$ has a zero in column $\mathbf{c}_i$.) This implies that the signs are incorrect for all nonzero entries in columns $\mathbf{c}_1, \ldots, \mathbf{c}_{i-1}$ within any row $\mathbf{r}_u$ in $S(\mathbf{c}_i)$ that shares a trivial 2×2 with any row $\mathbf{r}_t$ of $S(\mathbf{c}_i)$ such that $w\left(a_1^{t,i}, \ldots, a_{i-1}^{t,i}\right) \equiv 0$ modulo 2 and $\mathbf{G}(t,i)$ is nonzero. ($\mathbf{G}(u,i) = 0$, so we need not consider the sign of this entry.)

Hence, to restore the correct signs of nonzero entries located in columns $\mathbf{c}_1, \ldots, \mathbf{c}_{i-1}$ within any row $\mathbf{r}_u$ of $S(\mathbf{c}_i)$ that shares a trivial 2×2 with $\mathbf{r}_t$ (where $\mathbf{r}_t$ is in $S(\mathbf{c}_i)$, $w\left(a_1^{t,i}, \ldots, a_{i-1}^{t,i}\right) \equiv$ 0 modulo 2, and $\mathbf{G}(t,i)$ is nonzero), simply negate the row $\mathbf{r}_u$. If i is odd, this is equivalent to negating all rows $\mathbf{r}_u$ in $S(\mathbf{c}_i)$ such that $w\left(a_1^{u,i}, \ldots, a_{i-1}^{u,i}\right) \equiv 1$ modulo 2 and $\mathbf{G}(u,i) = 0$; if i is even, this is equivalent to negating all rows $\mathbf{r}_u$ in $S(\mathbf{c}_i)$ such that $w\left(a_1^{u,i}, \ldots, a_{i-1}^{u,i}\right) \equiv 0$ modulo 2 and $\mathbf{G}(u,i) = 0$.

In summary, the instance-negation phase of the algorithm for a set $S(\mathbf{c}_i)$ ruined the standard form of certain entries within columns $\mathbf{c}_1, \ldots, \mathbf{c}_{i-1}$, but it retained/gave the correct signs to entries in columns $\mathbf{c}_1, \ldots, \mathbf{c}_i$ within rows $\mathbf{r}_t$ in $S(\mathbf{c}_i)$ where $w\left(a_1^{t,i}, \ldots, a_{i-1}^{t,i}\right) \equiv 0$ modulo 2 and where $\mathbf{G}(t,i)$ is nonzero. The first step of the correction phase of the algorithm then corrected the ruined signs of entries located in columns $\mathbf{c}_1, \ldots, \mathbf{c}_{i-1}$ and corrected the initially incorrect sign on the entry in column $\mathbf{c}_i$ within rows $\mathbf{r}_u$ of $S(\mathbf{c}_i)$ where $w\left(a_1^{u,i}, \ldots, a_{i-1}^{u,i}\right) \equiv 1$ modulo 2 and where $\mathbf{G}(u,i)$ is nonzero (these are the rows $\mathbf{r}_u$ of $S(\mathbf{c}_i)$ that share an Alamouti 2×2 with such above-described rows $\mathbf{r}_t$). The second step of the correction phase then corrected the signs of entries located in columns $\mathbf{c}_1, \ldots, \mathbf{c}_{i-1}$ within the rows $\mathbf{r}_u$ of $S(\mathbf{c}_i)$ that contain a zero in column $\mathbf{c}_i$ (these are the rows $\mathbf{r}_u$ of $S(\mathbf{c}_i)$ that share a trivial 2×2 with such above-described rows $\mathbf{r}_t$). Since these two steps account for all nonzero entries in columns $\mathbf{c}_1, \ldots, \mathbf{c}_{i-1}$ within the rows of $S(\mathbf{c}_i)$ whose signs were made incorrect during the instance-negation phase of this algorithm, we have corrected the signs of all entries in columns $\mathbf{c}_1, \ldots, \mathbf{c}_{i-1}$ within the rows of $S(\mathbf{c}_i)$, as well as providing the correct signs in the nonzero entries of $\mathbf{c}_i$ within the rows of $S(\mathbf{c}_i)$. Hence, all entries in columns $\mathbf{c}_1, \ldots, \mathbf{c}_i$ within $S(\mathbf{c}_i)$ are now in standard form.

We repeat this entire algorithm on every 2×2 set with respect to column $\mathbf{c}_i$. Part 3) of Lemma 3.6 implies that after completing this algorithm on all 2×2 sets with respect to $\mathbf{c}_i$, all columns $\mathbf{c}_1, \ldots, \mathbf{c}_i$ are in standard form. Therefore our induction is complete. □

We note that for a given m, Theorem 3.7 implies that maximum rate, minimum delay CODs with $2m - 1$ columns are unique up to equivalence operations. This result is of theoretical interest, and it also has implications for practical implementations.

Example 3.8: This example illustrates some key steps in the execution of the standard form algorithm presented in the proof of Theorem 3.7 . Recall the $[15, 5, 10]$ COD $\mathbf{G}_2$ from Example 3.5. Consider $S_{\mathbf{r}_1}(\mathbf{c}_3)$, the 2×2 set with respect to column $\mathbf{c}_3$ using the first row $\mathbf{r}_1$ of $\mathbf{G}_2$ as the seed row; this set is displayed as the first four rows of $\mathbf{G}_2$. One can confirm that all entries within columns $\mathbf{c}_1$ and $\mathbf{c}_2$ are in standard form, and the entries within $\mathbf{c}_3$ that are *not* in $S_{\mathbf{r}_1}(\mathbf{c}_3)$ are also in standard form.

To put the entries within $S_{\mathbf{r}_1}(\mathbf{c}_3)$ in column $\mathbf{c}_3$ in standard form, we must negate all instances of variables that lie within column $\mathbf{c}_3$ within a row $\mathbf{r}_t$ in $S_{\mathbf{r}_1}(\mathbf{c}_3)$ that has $w\left(a_1^{t,3}, a_2^{t,3}\right) \equiv$ 0 modulo 2. Row $\mathbf{r}_1$ is the only such a row, hence we must negate all instances of z_1. The resulting matrix is shown below as $\mathbf{G}_2'$, and this completes the instance-negation phase of the algorithm for the specific set $S(\mathbf{c}_3)$.

$$\mathbf{G}_2' = \left(\begin{array}{ccccc}
z_2^* & -z_5^* & z_1^* & 0 & 0 \\
0 & z_1 & z_5 & -z_6 & -z_7 \\
-z_1 & 0 & z_2 & -z_3 & -z_4 \\
z_5 & z_2 & 0 & z_8 & z_9 \\
\hline
z_6 & z_3 & z_8 & 0 & z_{10} \\
0 & z_8^* & -z_3^* & -z_2^* & 0 \\
z_8^* & 0 & -z_6^* & -z_5^* & 0 \\
z_3^* & -z_6^* & 0 & -z_1^* & 0 \\
\hline
z_7 & z_4 & z_9 & -z_{10} & 0 \\
0 & z_9^* & -z_4^* & 0 & -z_2^* \\
z_9^* & 0 & -z_7^* & 0 & -z_5^* \\
z_4^* & -z_7^* & 0 & 0 & -z_1^* \\
\hline
z_{10}^* & 0 & 0 & z_7^* & -z_6^* \\
0 & z_{10}^* & 0 & z_4^* & -z_3^* \\
0 & 0 & z_{10}^* & z_9^* & -z_8^*
\end{array}\right).$$

The next matrix $\mathbf{G}_2''$ reflects the first step of the correction phase, where any row that shares a relevant Alamouti 2×2 with row $\mathbf{r}_1$ is corrected (namely $\mathbf{r}_2$ and $\mathbf{r}_3$ are negated); for the second step of the correction phase, where any row that shares a relevant trivial 2×2 with $\mathbf{r}_1$ is corrected, no action is taken because no such row exists in this example.

$$\mathbf{G}_2'' = \left(\begin{array}{ccccc}
z_2^* & -z_5^* & z_1^* & 0 & 0 \\
0 & -z_1 & -z_5 & z_6 & z_7 \\
z_1 & 0 & -z_2 & z_3 & z_4 \\
z_5 & z_2 & 0 & z_8 & z_9 \\
\hline
z_6 & z_3 & z_8 & 0 & z_{10} \\
0 & z_8^* & -z_3^* & -z_2^* & 0 \\
z_8^* & 0 & -z_6^* & -z_5^* & 0 \\
z_3^* & -z_6^* & 0 & -z_1^* & 0 \\
\hline
z_7 & z_4 & z_9 & -z_{10} & 0 \\
0 & z_9^* & -z_4^* & 0 & -z_2^* \\
z_9^* & 0 & -z_7^* & 0 & -z_5^* \\
z_4^* & -z_7^* & 0 & 0 & -z_1^* \\
\hline
z_{10}^* & 0 & 0 & z_7^* & -z_6^* \\
0 & z_{10}^* & 0 & z_4^* & -z_3^* \\
0 & 0 & z_{10}^* & z_9^* & -z_8^*
\end{array}\right).$$

Hence, since we have completed the correction algorithm on the rows in $S_{\mathbf{r}_1}(\mathbf{c}_3)$ within column $\mathbf{c}_3$, and since we started with correct entries in all other entries within column $\mathbf{c}_3$ as well as correct entries within all rows of columns $\mathbf{c}_1$ and $\mathbf{c}_2$, columns $\mathbf{c}_1$, $\mathbf{c}_2$, and $\mathbf{c}_3$ are now in standard form. The entries in columns $\mathbf{c}_4$ and $\mathbf{c}_5$ are not necessarily in standard form.

IV. Decoding Delay of $[r_0, 2m, r_0 R_0]$ CODs

In this section, we prove that maximum rate CODs with $2m$ columns, m odd, cannot achieve the lower bound on decoding delay, and at best can achieve twice the lower bound. We also present some implications for the number of variables in such CODs.

Theorem 4.1: A maximum rate COD with $2m$ columns cannot achieve the lower bound of $r_0 = \binom{2m}{m-1}$ on decoding delay if m is odd.

Proof: Suppose for contradiction that it is possible to form an $[r_0, 2m, r_0R_0]$ COD $\mathbf{G}'$ where m is odd. Recall that for a COD with $2m-1$ or $2m$ columns, the maximum rate is $\frac{m+1}{2m}$ and the lower bound on decoding delay is $\binom{2m}{m-1}$. Then, as any subset of columns of a COD forms another COD, the first $2m-1$ columns of $\mathbf{G}'$ themselves form a COD that is still of maximum rate and that still achieves the lower bound on delay. So, by Theorem 3.7, since the first $2m-1$ columns of $\mathbf{G}'$ form a $[r_0, 2m-1, r_0R_0]$ COD, we can perform equivalence operations on $\mathbf{G}'$ so that its first $2m-1$ columns are in standard form. We refer to the submatrix consisting of these first $2m-1$ columns as $\mathbf{G}$, and we assume hereafter that $\mathbf{G}'$ has undergone appropriate equivalence operations so that the submatrix $\mathbf{G}$ is in standard form.

Consider the following algorithm that selects $2m$ rows of $\mathbf{G}'$ such that for each $1 \leq i \leq 2m-1$, row $\mathbf{r}_i$ and row $\mathbf{r}_{i+1}$ share an Alamouti 2×2 over columns $\mathbf{c}_i$ and $\mathbf{c}_{2m}$. As $\mathbf{G}'$ is assumed to be maximum rate and to achieve the lower bound on delay, we can assume that every pattern of $m-1$ zeros appears in exactly one row [13], so we may consider a row with the following zero pattern:

$$\mathbf{r}_1 : z \quad 0 \quad z \quad 0 \quad \ldots \quad z \quad 0 \quad z \quad z$$

where z represents any nonzero variable (positive or negative, conjugated or nonconjugated).

Then, row $\mathbf{r}_2$ is selected as the row that shares an Alamouti 2×2 with row $\mathbf{r}_1$ over columns $\mathbf{c}_1$ and $\mathbf{c}_{2m}$. By Lemma 3.2, row $\mathbf{r}_2$ will be nonzero in columns $\mathbf{c}_1$ and $\mathbf{c}_{2m}$, while its remaining entries will be zero when the corresponding entry of $\mathbf{r}_1$ is nonzero and vice versa. Thus, $\mathbf{r}_2$ has the following form:

$$\mathbf{r}_2 : z \quad z \quad 0 \quad z \quad \ldots \quad 0 \quad z \quad 0 \quad z$$

where again z represents any nonzero variable.

In general, select $\mathbf{r}_{i+1}$ as the row that shares an Alamouti 2×2 with $\mathbf{r}_i$ over columns $\mathbf{c}_i$ and $\mathbf{c}_{2m}$. Such a row $\mathbf{r}_{i+1}$ must exist to ensure the orthogonality of $\mathbf{G}'$, assuming that the entries of row $\mathbf{r}_i$ in columns $\mathbf{c}_i$ and $\mathbf{c}_{2m}$ are nonzero, which we will prove inductively. The base case uses the given $\mathbf{r}_1$, which is clearly nonzero in columns $\mathbf{c}_1$ and $\mathbf{c}_{2m}$. Now, assume that for some $1 \leq \ell < 2m$ the row $\mathbf{r}_\ell$ is nonzero in columns $\mathbf{c}_\ell$ and $\mathbf{c}_{2m}$. Now consider row $\mathbf{r}_{\ell+1}$, which will share an Alamouti 2×2 with row $\mathbf{r}_\ell$, as required for the orthogonality of columns $\mathbf{c}_\ell$ and $\mathbf{c}_{2m}$. By Lemma 3.2, since $\mathbf{r}_\ell$ is nonzero in columns $\mathbf{c}_\ell$ and $\mathbf{c}_{2m}$, the new row $\mathbf{r}_{\ell+1}$ will also be nonzero in columns $\mathbf{c}_\ell$ and $\mathbf{c}_{2m}$. However, we must further show that $\mathbf{r}_{\ell+1}$ is nonzero in column $\mathbf{c}_{\ell+1}$. If $\ell+1$ is odd, then $\mathbf{r}_1$ is nonzero in column $\mathbf{c}_{\ell+1}$. The formation of rows $\mathbf{r}_j$ for $j \leq \ell$ has not involved Alamouti 2×2's with column $\mathbf{c}_{\ell+1}$, so Lemma 3.2 shows that the entries of these rows within column $\mathbf{c}_{\ell+1}$ will have alternated between nonzero and zero ℓ times. Since $\ell+1$ is odd, by row $\mathbf{r}_{\ell+1}$ this position will have alternated back to being nonzero. The case where $\ell+1$ is even follows similarly. Hence, by the principle of mathematical induction, we have shown that for any $1 \leq i < 2m$, row $\mathbf{r}_i$ is nonzero in columns $\mathbf{c}_i$ and $\mathbf{c}_{2m}$. Hence, for each $1 \leq i < 2m$, the proposed row $\mathbf{r}_{i+1}$ must exist.

Throughout this algorithm, each column $\mathbf{c}_i$, for $1 \leq i \leq 2m-1$, is paired once with column $\mathbf{c}_{2m}$ during the creation of an Alamouti 2×2 shared between rows $\mathbf{r}_i$ and $\mathbf{r}_{i+1}$. Given the initial row $\mathbf{r}_1$, these $2m-1$ pairing produce rows $\mathbf{r}_2, \ldots, \mathbf{r}_{2m}$, in order. Lemma 3.2 shows that when moving from $\mathbf{r}_i$ to $\mathbf{r}_{i+1}$, the entries in columns $\mathbf{c}_i$ and $\mathbf{c}_{2m}$ remain nonzero, while all other column positions change from zero to nonzero or vice versa. Thus, over the course of the $2m-1$ pairings, every column position changes from zero to nonzero (or vice versa) exactly $2m-2$ times. Thus, after $2m-1$ pairings, we have changed each column position an even number of times, producing a row $\mathbf{r}_{2m}$ that has exactly the same pattern of zeros as row $\mathbf{r}_1$. Since a maximum rate COD that achieves the lower bound on delay has exactly one row with any specific pattern of zeros [13], rows $\mathbf{r}_1$ and $\mathbf{r}_{2m}$ must be the same row, while rows $\mathbf{r}_1, \ldots, \mathbf{r}_{2m-1}$ are distinct.

Next, we must look at the signs of the entries of the selected rows in columns $\mathbf{c}_i$ for $1 \leq i \leq 2m-1$, which is possible as these first $2m-1$ columns are in standard form. For consistency of notation, rearrange the rows of $\mathbf{G}'$ so that the $2m-1$ distinct rows $\mathbf{r}_1, \ldots, \mathbf{r}_{2m-1}$, selected by the algorithm are the first $2m-1$ rows of $\mathbf{G}'$, in order. Then, we need to examine the sign of the ith entries in rows $\mathbf{r}_i$ and $\mathbf{r}_{i+1}$ for each $1 \leq i \leq 2m-1$; the case of $i = 2m-1$ should be interpreted as examining the sign of the $(2m-1)^{th}$ entries of rows $\mathbf{r}_{2m-1}$ and $\mathbf{r}_1$, as the row $\mathbf{r}_{2m}$ selected by the algorithm was shown to be equal to the row $\mathbf{r}_1$ selected by the algorithm, which is currently listed as the first row of $\mathbf{G}'$.

Note that when ignoring the last column of rows $\mathbf{r}_i$ and $\mathbf{r}_{i+1}$, these shortened rows, denoted as $\mathbf{r}_i^{-1}$ and $\mathbf{r}_{i+1}^{-1}$, respectively, are rows in the valid $[r_0, 2m-1, r_0R_0]$ COD $\mathbf{G}$ that is in standard form. We have $\tilde{\mathbf{r}}_i^{-1} - \tilde{\mathbf{r}}_{i+1}^{-1} = 111\ldots101\ldots111$, where the only 0 is in the i^{th} position. It can be shown that $\tilde{\mathbf{r}}_i^{-1} - \tilde{\mathbf{r}}_{i+1}^{-1} = \sum_{k,k\neq i}^{2m-1} \mathbf{v}_{i,k}$. Hence, as in the proof of Lemma 3.2, if $i \geq 2$ and if the vector expansion of $\tilde{\mathbf{r}}_i^{-1}$ over $\mathbf{V}_i$ is $\tilde{\mathbf{r}}_i^{-1} = a_1^{i,i}\mathbf{v}_{i,1} + a_2^{i,i}\mathbf{v}_{i,2} + \cdots + a_i^{i,i}\mathbf{v}_{i,i} + \cdots + a_{2m-1}^{i,i}\mathbf{v}_{i,2m-1}$, then we can write the vector expansion of $\tilde{\mathbf{r}}_{i+1}^{-1}$ over $\mathbf{V}_i$ as $\tilde{\mathbf{r}}_{i+1}^{-1} = \bar{a}_1^{i,i}\mathbf{v}_{i,1} + \bar{a}_2^{i,i}\mathbf{v}_{i,2} + \cdots + a_i^{i,i}\mathbf{v}_{i,i} + \cdots + \bar{a}_{2m-1}^{i,i}\mathbf{v}_{i,2m-1}$, where $\bar{a}$ is the binary compliment of a.

So, for $i \geq 2$, the first $i-1$ coefficients of these vector expansions are exactly opposite. If i is even (respectively, odd), then we are complementing an odd (respectively, even) number $i-1$ of binary coefficients, and it follows that $w\left(a_1^{i,i}, \ldots, a_{i-1}^{i,i}\right) \not\equiv w\left(a_1^{i+1,i}, \ldots, a_{i-1}^{i+1,i}\right)$ modulo 2 (respectively, $w\left(a_1^{i,i}, \ldots, a_{i-1}^{i,i}\right) \equiv w\left(a_1^{i+1,i}, \ldots, a_{i-1}^{i+1,i}\right)$ modulo 2). Thus, the signs of entries $\mathbf{G}(i,i)$ and $\mathbf{G}(i+1,i)$ are different (respectively, the same) since $\mathbf{G}$ is assumed to be in standard form. It then follows that $\mathbf{G}'(i,2m)$ and $\mathbf{G}'(i+1,2m)$ must have the same (respectively, different) signs to maintain the orthogonality of the Alamouti 2×2 in the expanded matrix $\mathbf{G}'$. If $i = 2m-1$, this should be interpreted to mean that $\mathbf{G}'(2m-1,2m)$ and $\mathbf{G}'(1,2m)$ must have different signs.

If $i = 1$, (which must be considered separately as the definition of standard form for $\mathbf{c}_1$ differs from the general definition for $\mathbf{c}_i, i \geq 2$), the general result for i odd still holds as the signs of $\mathbf{G}(1,1)$ and $\mathbf{G}(2,1)$ are the same, since the standard form of column $\mathbf{c}_1$ indicates that all nonzero entries in $\mathbf{c}_1$ have positive signs. So, $\mathbf{G}'(1,2m)$ and $\mathbf{G}'(2,2m)$ must have different signs to maintain the orthogonality of $\mathbf{G}'$.

Now, consider the nonzero entries $\mathbf{G}'(1,2m), \mathbf{G}'(2,2m), \ldots, \mathbf{G}'(2m-1,2m)$. By the work above, if i is odd, $1 \leq i \leq 2m-3$, the entry $\mathbf{G}'(i,2m)$ and the entry $\mathbf{G}'(i+1,2m)$ will have opposite signs. If i is even, for $2 \leq i \leq 2m-2$, the entry $\mathbf{G}'(i,2m)$ and the entry $\mathbf{G}'(i+1,2m)$ will have the same sign. Hence, we can consider a "path" through the $2m-1$ entries $\mathbf{G}'(1,2m), \mathbf{G}'(2,2m), \ldots, \mathbf{G}'(2m-1,2m)$ that records the relationships of the signs of these entries. We begin with a "different sign" relationship, followed by a "same sign" relationship, and we continue to alternate until we end with a "same sign" relationship. (For example, if entry $\mathbf{G}'(1,2m)$ is positive, then $\mathbf{G}'(2,2m)$ is negative, $\mathbf{G}'(3,2m)$ is negative, $\mathbf{G}'(4,2m)$ is positive, $\mathbf{G}'(5,2m)$ is positive, and so on.) This path contains $m-1$ "different sign" relationships and $m-1$ "same sign" relationships, so if we begin with a positive (resp., negative) sign on entry $\mathbf{G}'(1,2m)$, this sign will change $m-1$ times and remain the same $m-1$ times, implying that if m is odd, then the sign on $\mathbf{G}'(2m-1,2m)$ will again be positive (respectively, negative). However, this contradicts our above work showing that entries $\mathbf{G}'(1,2m)$ and $\mathbf{G}'(2m-1,2m)$ must have different signs.

So, for m odd, we began with a valid $[r_0, 2m, r_0R_0]$ COD $\mathbf{G}'$, performed equivalence operations on $\mathbf{G}'$, and then showed that the version of $\mathbf{G}'$ after undergoing equivalence operations is no longer a valid COD. Thus, our original assumption that there exists some valid $[r_0, 2m, r_0R_0]$ COD where m is odd must be false. Hence, any maximum rate COD with $2m$ columns for m odd cannot achieve the lower bound on decoding delay. □

Theorem 4.1 can be viewed to imply that a maximum rate, minimum decoding delay $[r_0, 2m-1, r_0R_0]$ COD, where m is odd, cannot be padded to form a maximum rate, minimum decoding delay $[r_0, 2m, r_0R_0]$ COD. In fact, the delay must be doubled when moving from this $2m-1$ column case to the $2m$ column case, as shown in the following corollary.

Corollary 4.2: The minimum achievable decoding delay for a maximum rate COD with $2m$ columns, m odd, is $2r_0 = 2\binom{2m}{m-1}$.

Proof: It follows directly from [13, Theorem 3.1] and Theorem 4.1 above that the best possible delay for such a code is twice the lower bound, $2r_0 = 2\binom{2m}{m-1}$. The fact that this delay is achievable relies on algorithms by other authors, as explained in the partial proof to [13, Conjecture 3.3]: Lu *et al.* have proven that their algorithm achieves this delay in this case [14], and other algorithms can also be observed to achieve this delay [7], [12].

Corollary 4.3: Suppose that $\mathbf{G}$ is a maximum rate complex orthogonal design with $2m-1$ or $2m$ columns. Then a lower bound on the number of variables required is $\frac{1}{2}\binom{2m}{m}$. This lower bound can be achieved when the number of columns is congruent to 0, 1, or 3 modulo 4. When the number of columns is congruent to 2 modulo 4, the best achievable bound is $\binom{2m}{m}$.

Proof: This follows directly from [13, Corollary 4.1] and Theorem 4.1 above. □

V. CONCLUSION

This paper closes the final case in determining the minimum achievable decoding delay for maximum rate CODs and their associated COSTBCs. We have shown that if a maximum rate COD $\mathbf{G}$ has n congruent to 2 modulo 4 columns, then it can at best achieve a delay of twice the lower bound. This final case required machinery beyond that which was employed to prove the other cases. The bases $\mathbf{V}_i$ provide new techniques for analyzing CODs, and we hope that our linear algebraic machinery will be helpful to other researchers in this area.

Our work also implied that there is exactly one equivalence class of maximum rate, minimum decoding delay CODs with $2m-1$ columns.

We can now summarize all known results concerning the minimum decoding delay of maximum rate CODs. The minimum achievable decoding delay for a maximum rate COD with $2m$ columns with m even or any $2m-1$ columns is $r_0 = \binom{2m}{m-1}$, while the minimum achievable decoding delay for a maximum rate COD with $2m$ columns with m odd is $2r_0 = 2\binom{2m}{m-1}$. In other words, for $n \not\equiv 2$ modulo 4, we say that a maximum rate, minimum decoding delay COD has parameters $[r_0, n, r_0R_0]$, and for $n \equiv 2$ modulo 4, a maximum rate, minimum decoding delay COD has parameters $[2r_0, n, 2r_0R_0]$.

This paper completes the study of the minimum decoding delay of maximum rate CODs, while again highlighting the quick growth of the decoding delay as the number of columns increases. Since the maximum rate approaches $1/2$ as the number of columns increases, it will be important to determine the minimum decoding delay for rate $1/2$ CODs to see if the small reduction in rate would result in a large improvement in decoding delay.

In this paper, we restricted our attention to CODs without linear processing (LP), consistent with the previously cited work on maximum rate [7] and minimum decoding delay [13]. The same questions of maximum rate and minimum decoding delay are also important for CODs with LP. Though some work has been done concerning the existence and optimal parameters of CODs with LP [1], [17]–[19], it remains open whether the optimal rate and delay for generalized CODs with LP are as they are for CODs without LP.

VI. ACKNOWLEDGEMENTS

The authors would like to thank J. Pollack for his work on related results and A. Dorsk and A. Kalcic for their related work on signed graphs, which inspired the proof techniques in Section IV. The authors also thank the reviewers for their careful comments which helped to greatly improve this version of the paper.

APPENDIX

This Appendix contains the proofs of Lemmas 3.2 and 3.6.

1) Proof of Lemma 3.2: We assume throughout that $n = 2m-1$, as the relevant cases where $n = 2m$ are simpler and follow similarly.

Part 1) Suppose that two distinct rows $\mathbf{r}_s$ and $\mathbf{r}_t$ of an $[r_0, 2m-1, r_0R_0]$ COD $\mathbf{G}$ share an Alamouti 2×2 over columns $\mathbf{c}_h$ and $\mathbf{c}_i$, say containing z_a (up to sign) and z_a^* (up to sign) for some $1 \leq a \leq r_0 R_0$. We can perform equivalence operations on $\mathbf{G}$ to obtain the $\mathbf{B}_a$ submatrix [7] and in particular to transform $\mathbf{r}_s$ and $\mathbf{r}_t$ into rows $\mathbf{r}'_s$ and $\mathbf{r}'_t$ of $\mathbf{B}_a$. Similarly, we will write that columns $\mathbf{c}_h$ and $\mathbf{c}_i$ have been transformed into columns $\mathbf{c}'_h$ and $\mathbf{c}'_i$. Since $\mathbf{r}_s$ and $\mathbf{r}_t$ share an Alamouti 2×2 over columns $\mathbf{c}_h$ and $\mathbf{c}_i$, $\mathbf{r}'_s$ and $\mathbf{r}'_t$ likewise share an Alamouti 2×2 over columns $\mathbf{c}'_h$ and $\mathbf{c}'_i$. Moreover, we can assume $\mathbf{r}'_s$ will be among the first m rows of $\mathbf{B}_a$, $\mathbf{r}'_t$ will be among the last $m-1$ rows of $\mathbf{B}_a$.

Since $\mathbf{M}_a$ and $-\mathbf{M}_a^H$ contain no zero entries [7], the $m-1$ zeros of $\mathbf{r}'_s$ occur in columns in which $\mathbf{r}'_t$ is nonzero, the $m-2$ zeros of $\mathbf{r}'_t$ occur in column positions in which $\mathbf{r}'_s$ is nonzero, and both rows are simultaneously nonzero in exactly columns $\mathbf{c}'_h$ and $\mathbf{c}'_i$. Thus, the original rows $\mathbf{r}_s$ and $\mathbf{r}_t$ were simultaneously nonzero exactly in columns $\mathbf{c}_h$ and $\mathbf{c}_i$ and simultaneously zero in no column position. This proves the forward direction.

Now suppose that $\mathbf{r}_u$ and $\mathbf{r}_v$ are two rows of $\mathbf{G}$ such that $\mathbf{r}_u$ and $\mathbf{r}_v$ are simultaneously nonzero in exactly two columns and simultaneously zero in no column. Thus, there are $2m-3$ columns in which exactly one of $\mathbf{r}_u$ or $\mathbf{r}_v$ must be zero. Since all rows in a maximum rate COD with $2m-1$ columns have either $m-1$ or $m-2$ zeros [7], this implies that one of these rows, say $\mathbf{r}_u$, must have $m-1$ zeros and the other, say $\mathbf{r}_v$, must have $m-2$ zeros.

As the $m-1$ zeros of $\mathbf{r}_u$ and the $m-2$ zeros of $\mathbf{r}_v$ overlap in no columns, there exist suitable column rearrangements that convert $\mathbf{r}_u$ and $\mathbf{r}_v$ into rows $\mathbf{r}'_u$ and $\mathbf{r}'_v$ such that $\mathbf{r}'_u$ and $\mathbf{r}'_v$ each have a nonzero in column $\mathbf{c}_1$, $\mathbf{r}'_u$ has $m-1$ zeros and $\mathbf{r}'_v$ has $m-1$ nonzeros in columns $\mathbf{c}_2, \ldots, \mathbf{c}_m$, $\mathbf{r}'_u$ and $\mathbf{r}'_v$ are again each nonzero in column $\mathbf{c}_{m+1}$, and $\mathbf{r}'_u$ has $m-2$ nonzeros and $\mathbf{r}'_v$ has $m-2$ zeros in columns $\mathbf{c}_{m+2}, \ldots, \mathbf{c}_{2m-1}$.

Suppose the intersection of row $\mathbf{r}'_u$ and column $\mathbf{c}_1$ contains an instance of the variable z_a. An $[r_0, 2m-1, r_0 R_0]$ COD can contain only one row with a particular pattern of zeros [13], so the structure of $\mathbf{r}'_u$ implies that it is a row within the $\mathbf{B}_a$ submatrix (up to conjugation and sign). Similarly, the structure of $\mathbf{r}'_v$ shows that it is also a row within $\mathbf{B}_a$ (again, up to conjugation and sign). This implies that the intersection of row $\mathbf{r}'_v$ and column $\mathbf{c}_{m+1}$ contains an instance of z_a. It then follows that $\mathbf{r}'_u$ and $\mathbf{r}'_v$ share an Alamouti 2×2 over columns $\mathbf{c}_1$ and $\mathbf{c}_{m+1}$. Hence, the original rows $\mathbf{r}_u$ and $\mathbf{r}_v$ must share an Alamouti 2×2 over the two columns in which they are simultaneously nonzero, thus proving the reverse direction.

In the case of $n = 2m-1$ only, we can restate this result in terms of the basis $\mathbf{V}_i$ of length $2m-1$ vectors: Two rows $\mathbf{r}_s$ and $\mathbf{r}_t$ share an Alamouti 2×2 over columns $\mathbf{c}_h$ and $\mathbf{c}_i$ if and only if $\tilde{\mathbf{r}}_s - \tilde{\mathbf{r}}_t = \mathbf{v}_{h,i}$ (under binary subtraction). This follows directly as $\mathbf{v}_{h,i}$ has a 1 in all positions except for 0's in positions h and i. So, the vector expansion of $\tilde{\mathbf{r}}_s$ over $\mathbf{V}_i$ will only differ from the vector expansion of $\tilde{\mathbf{r}}_t$ over $\mathbf{V}_i$ in the coefficient on $\mathbf{v}_{h,i} = \mathbf{v}_{i,h}$. Hence, if $i > h$ and if the vector expansion of $\tilde{\mathbf{r}}_s$ over $\mathbf{V}_i$ is $\tilde{\mathbf{r}}_s = a_1^{s,i}\mathbf{v}_{i,1} + a_2^{s,i}\mathbf{v}_{i,2} + \cdots + a_h^{s,i}\mathbf{v}_{i,h} + \cdots + a_i^{s,i}\mathbf{v}_{i,i} + \cdots + a_{2m-1}^{s,i}\mathbf{v}_{i,2m-1}$, then the vector expansion of $\tilde{\mathbf{r}}_t$ over $\mathbf{V}_i$ can be written as $\tilde{\mathbf{r}}_t = a_1^{s,i}\mathbf{v}_{i,1} + a_2^{s,i}\mathbf{v}_{i,2} + \cdots + a_{h-1}^{s,i}\mathbf{v}_{i,h-1} + \bar{a}_h^{s,i}\mathbf{v}_{i,h} + a_{h+1}^{s,i}\mathbf{v}_{i,h+1} + \cdots + a_i^{s,i}\mathbf{v}_{i,i} + \cdots + a_{2m-1}^{s,i}\mathbf{v}_{i,2m-1}$, where $\bar{a}$ is the binary compliment of a. So, we have

$$w\left(a_1^{t,i}, \ldots, a_h^{t,i}, \ldots, a_{i-1}^{t,i}\right) = w\left(a_1^{s,i}, \ldots, \bar{a}_h^{s,i}, \ldots, a_{i-1}^{s,i}\right)$$

hence

$$w\left(a_1^{t,i}, \ldots, a_h^{t,i}, \ldots, a_{i-1}^{t,i}\right) \not\equiv w\left(a_1^{s,i}, \ldots, a_h^{s,i}, \ldots, a_{i-1}^{s,i}\right)$$

modulo 2. This implies the standard form signs of entries $\mathbf{G}(s,i)$ and $\mathbf{G}(t,i)$ are different.

If instead we look at the vector expansions of $\tilde{\mathbf{r}}_s$ and $\tilde{\mathbf{r}}_t$ over $\mathbf{V}_h$ $(h < i)$, they will again only differ in the coefficient on $\mathbf{v}_{h,i} = \mathbf{v}_{i,h}$. So, if $h > 1$, the first $h-1$ coefficients of these vector expansions over $\mathbf{V}_h$ are identical, hence $w\left(a_1^{s,h}, \ldots, a_{h-1}^{s,h}\right) \equiv w\left(a_1^{t,h}, \ldots, a_{h-1}^{t,h}\right)$ modulo 2 and the standard form signs of entries $\mathbf{G}(s,h)$ and $\mathbf{G}(t,h)$ are the same. If $h = 1$, the definition of standard form for column $\mathbf{c}_1$ indicates that the signs of entries $\mathbf{G}(s,h)$ and $\mathbf{G}(t,h)$ are again the same.

Part 2) Suppose that two distinct rows $\mathbf{r}_s$ and $\mathbf{r}_t$ of an $[r_0, 2m-1, r_0 R_0]$ COD $\mathbf{G}$ share a trivial 2×2 over columns $\mathbf{c}_h$ and $\mathbf{c}_i$ containing two instances of the variable z_a for some $1 \leq a \leq r_0 R_0$. Then, use equivalence operations to form the $\mathbf{B}_a$ submatrix [7] and in particular to transform $\mathbf{r}_s$ and $\mathbf{r}_t$ into rows $\mathbf{r}'_s$ and $\mathbf{r}'_t$ of $\mathbf{B}_a$. Similarly, we will write that columns $\mathbf{c}_h$ and $\mathbf{c}_i$ have been transformed into columns $\mathbf{c}'_h$ and $\mathbf{c}'_i$.

We will assume that $\mathbf{r}'_s$ and $\mathbf{r}'_t$ lie within the first m rows of $\mathbf{B}_a$, hence, each contain z_a. The case where $\mathbf{r}'_s$ and $\mathbf{r}'_t$ lie within the last $m-1$ rows of $\mathbf{B}_a$ and each contain z_a^* follows similarly. As the $\mathbf{M}_a$ submatrix contains no zero entries [7], both $\mathbf{r}'_s$ and $\mathbf{r}'_t$ are simultaneously nonzero in columns $\mathbf{c}_{m+1}, \ldots, \mathbf{c}_{2m-1}$. In columns $\mathbf{c}_1, \ldots, \mathbf{c}_m$, both $\mathbf{r}'_s$ and $\mathbf{r}'_t$ contain $m-1$ zeros and one instance of z_a. These instances of z_a appear in different columns by our definition of a COD. Thus, rows $\mathbf{r}'_s$ and $\mathbf{r}'_t$ (resp., rows $\mathbf{r}_s$ and $\mathbf{r}_t$) are simultaneously zero or simultaneously nonzero in all columns except for within $\mathbf{c}'_h$ and $\mathbf{c}'_i$ (resp., $\mathbf{c}_h$ and $\mathbf{c}_i$), the columns under which they share a trivial 2×2. This proves the forward direction.

Now suppose that $\mathbf{r}_u$ and $\mathbf{r}_v$ are two rows of $\mathbf{G}$ such that $\mathbf{r}_u$ and $\mathbf{r}_v$ are either simultaneously zero or simultaneously nonzero in all but two columns $\mathbf{c}_h$ and $\mathbf{c}_i$. By this assumption, the entries in column $\mathbf{c}_h$ within rows $\mathbf{r}_u$ and $\mathbf{r}_v$ cannot both be zero or both be nonzero, and similarly for $\mathbf{c}_i$. We will now show the stronger result that the entries in row $\mathbf{r}_u$ within columns $\mathbf{c}_h$ and $\mathbf{c}_i$ cannot both be zero or both be nonzero, and similarly for $\mathbf{r}_v$. For contradiction, suppose that $\mathbf{r}_u$ contained an instance of zero in both $\mathbf{c}_h$ and $\mathbf{c}_i$. Our assumptions then imply that $\mathbf{r}_v$ contains a nonzero in both $\mathbf{c}_h$ and $\mathbf{c}_i$. Recall that each row in a maximum rate COD with $2m-1$ columns contains either $m-1$ or $m-2$ zeros [7], and suppose first that $\mathbf{r}_u$ contains $m-1$ total zeros. Then, in addition to the zeros in columns $\mathbf{c}_h$ and $\mathbf{c}_i$, $\mathbf{r}_u$ contains $m-3$ zeros within columns $\mathbf{c}_j$, $j = 1, \ldots, h-1, h+1, \ldots, i-1, i+1, \ldots, 2m-1$. Our assumptions then imply that $\mathbf{r}_v$ must also contain exactly $m-3$ zeros within columns $\mathbf{c}_j$, $j = 1, \ldots, h-1, h+1, \ldots, i-1, i+1, \ldots, 2m-1$. However, $\mathbf{r}_v$ is already assumed to be nonzero in the two remaining columns $\mathbf{c}_h$ and $\mathbf{c}_i$, which would leave $\mathbf{r}_v$ with only $m-3$ zeros, contradicting the fact that each row must contain either $m-1$ or $m-2$ zeros [7]. Similarly, we see that if $\mathbf{r}_u$ contains $m-2$ total zeros, then $\mathbf{r}_v$ would contain $m-4$ total zeros, which is also impossible. Thus, we have shown through contradiction that the entries in row $\mathbf{r}_u$ within columns $\mathbf{c}_h$ and

$\mathbf{c}_i$ cannot both be zero or both be nonzero, and similarly for $\mathbf{r}_v$. This implies directly that $\mathbf{r}_u$ and $\mathbf{r}_v$ contain the same number of zeros. We will consider the case where $\mathbf{r}_u$ and $\mathbf{r}_v$ both contain $m-1$ zeros, as the case where they both contain $m-2$ zeros follows similarly. We may now rearrange the columns of $\mathbf{G}$ so that rows $\mathbf{r}_u$ and $\mathbf{r}_v$ are transformed into rows $\mathbf{r}'_u$ and $\mathbf{r}'_v$ such that $\mathbf{r}'_u$ is zero and $\mathbf{r}'_v$ is nonzero in column $\mathbf{c}_1$, $\mathbf{r}'_u$ is nonzero and $\mathbf{r}'_v$ is zero in column $\mathbf{c}_2$, and $\mathbf{r}'_u$ and $\mathbf{r}'_v$ are simultaneously zero in the $m-2$ columns $\mathbf{c}_3, \ldots, \mathbf{c}_m$ and simultaneously nonzero in the $m-1$ columns $\mathbf{c}_{m+1}, \ldots, \mathbf{c}_{2m-1}$.

Suppose that the intersection of row $\mathbf{r}'_u$ and column $\mathbf{c}_2$ contains an instance of z_a. As an $[r_0, 2m-1, r_0R_0]$ COD can contain only one row with a particular pattern of zeros [13], the structure of row $\mathbf{r}'_u$ implies that it is a row within the $\mathbf{B}_a$ submatrix (up to conjugation and sign). Similarly, the structure of $\mathbf{r}'_v$ shows that it is also a row within the $\mathbf{B}_a$ submatrix (again up to conjugation and sign); hence the first entry of $\mathbf{r}'_v$ must contain an instance of z_a. It is now clear that $\mathbf{r}'_u$ and $\mathbf{r}'_v$ share a trivial 2×2 containing two instances of z_a over columns $\mathbf{c}_1$ and $\mathbf{c}_2$. This implies that the original rows $\mathbf{r}_u$ and $\mathbf{r}_v$ share a trivial 2×2 over the two columns in which they are not simultaneously zero or nonzero, and the reverse direction is proved.

In the case of $n = 2m-1$ only, this result can be restated in terms of the basis $\mathbf{V}_i$ as follows: Two rows $\mathbf{r}_s$ and $\mathbf{r}_t$ share a trivial 2×2 over columns $\mathbf{c}_h$ and $\mathbf{c}_i$ if and only if the binary difference $\tilde{\mathbf{r}}_s - \tilde{\mathbf{r}}_t = \bar{\mathbf{v}}_{i,h}$, where $\bar{\mathbf{v}}_{i,h}$ denotes the binary complement of $\mathbf{v}_{i,h}$. It then follows that $\tilde{\mathbf{r}}_s - \tilde{\mathbf{r}}_t = \sum_{k \neq h} \mathbf{v}_{i,k}$. So, if a basis vector $\mathbf{v}_{i,k}$, $k \neq h$, is used in the vector expansion of $\tilde{\mathbf{r}}_s$ over $\mathbf{V}_i$, then it is not used in the vector expansion of $\tilde{\mathbf{r}}_t$ over $\mathbf{V}_i$ and vice versa. However, if the basis vector $\mathbf{v}_{i,h}$ is (not) used in the expansion of $\tilde{\mathbf{r}}_s$ over $\mathbf{V}_i$, it is also (not) used in the expansion of $\tilde{\mathbf{r}}_t$ over $\mathbf{V}_i$, and vice versa.

So, using the vector expansion of $\tilde{\mathbf{r}}_s$ over $\mathbf{V}_i$ to write $\tilde{\mathbf{r}}_s = a_1^{s,i}\mathbf{v}_{i,1} + a_2^{s,i}\mathbf{v}_{i,2} + \cdots + a_h^{s,i}\mathbf{v}_{i,h} + \cdots + a_{2m-1}^{s,i}\mathbf{v}_{i,2m-1}$, we see that $\tilde{\mathbf{r}}_s - \tilde{\mathbf{r}}_t = \sum_{k \neq h} \mathbf{v}_{i,k}$ implies that the vector expansion of $\tilde{\mathbf{r}}_t$ over $\mathbf{V}_i$ can be written as $\tilde{\mathbf{r}}_t = \bar{a}_1^{s,i}\mathbf{v}_{i,1} + \bar{a}_2^{s,i}\mathbf{v}_{i,2} + \cdots + \bar{a}_{h-1}^{s,i}\mathbf{v}_{i,h-1} + a_h^{s,i}\mathbf{v}_{i,h} + \bar{a}_{h+1}^{s,i}\mathbf{v}_{i,h+1} + \cdots + \bar{a}_{2m-1}^{s,i}\mathbf{v}_{i,2m-1}$, where again $\bar{a}$ is the binary compliment of a.

If $i > h$ (so $i \geq 2$), then exactly $i-2$ of the first $i-1$ coefficients of the vector expansions of $\tilde{\mathbf{r}}_s$ and $\tilde{\mathbf{r}}_t$ are exactly opposite. Hence, if i is even, then we are complementing an even number $i-2$ of binary coefficients and simple algebra shows that $w\left(a_1^{s,i}, \ldots, a_{i-1}^{s,i}\right) \equiv w\left(a_1^{t,i}, \ldots, a_{i-1}^{t,i}\right)$ modulo 2. Similarly, if i is odd, $w\left(a_1^{s,i}, \ldots, a_{i-1}^{s,i}\right) \not\equiv w\left(a_1^{t,i}, \ldots, a_{i-1}^{t,i}\right)$ modulo 2. □

2) Proof of Lemma 3.6: Part 1) For $2 \leq i \leq 2m-1$, suppose that an instance of some variable z in column $\mathbf{c}_i$ within a row $\mathbf{r}_s$ of $S(\mathbf{c}_i)$ has the incorrect sign with respect to the standard form. We will consider the case where $w\left(a_1^{s,i}, \ldots, a_{i-1}^{s,i}\right) \equiv 0$ modulo 2, but the instance of z in $\mathbf{G}(s,i)$ is negative. The other case follows similarly.

It suffices to consider the rows $\mathbf{r}_t$ in $S(\mathbf{c}_i)$ that share an Alamouti 2×2 with $\mathbf{r}_s$ over column $\mathbf{c}_i$ and some $\mathbf{c}_h$, $h < i$, and we suppose that $\mathbf{G}(t,i)$ contains an instance of variable w. (In such a row that shares a trivial 2×2 with $\mathbf{r}_s$, we would have $\mathbf{G}(t,i) = 0$.) Then, $\mathbf{G}(s,h)$ must contain an instance of w and $\mathbf{G}(t,h)$ must contain an instance of z. Since column $\mathbf{c}_h$, $h < i$, is assumed to be in standard form, Lemma 3.2 indicates that entries $\mathbf{G}(s,h)$ and $\mathbf{G}(t,h)$ should have the same sign. Thus, to preserve orthogonality, since $\mathbf{G}(s,i)$ is assumed to be negative, $\mathbf{G}(t,i)$ must be positive. However, since we assumed $w\left(a_1^{s,i}, \ldots, a_{i-1}^{s,i}\right) \equiv 0$ modulo 2, the proof of Lemma 3.2 indicates that $w\left(a_1^{t,i}, \ldots, a_{i-1}^{t,i}\right) \equiv 1$ modulo 2, which would indicate that the standard form sign of $\mathbf{G}(t,i)$ is negative. So, $\mathbf{G}(t,i)$ has the incorrect sign with respect to standard form.

Continuing to chain together rows that share Alamouti 2×2's exhausts all of $S(\mathbf{c}_i)$, and we may conclude that all other entries in column $\mathbf{c}_i$ within a row of $S(\mathbf{c}_i)$ have the incorrect sign.

Part 2) For some $2 \leq i \leq 2m-1$, suppose that column $\mathbf{c}_i$ contains an instance of the variable z within a row $\mathbf{r}_s$ in $S(\mathbf{c}_i)$. Then, to maintain orthogonality of $\mathbf{G}$, this instance of z in entry $\mathbf{G}(s,i)$ appears within an Alamouti or trivial 2×2 with each entry $\mathbf{G}(s,h)$ for each $1 \leq h < i$. For a given h, the second row of the 2×2 contains an instance of z within column $\mathbf{c}_h$, and this row belongs to $S(\mathbf{c}_i)$ by the definition of such a 2×2 set. Since each variable appears exactly once per column, every instance of z within columns $\mathbf{c}_1, \ldots, \mathbf{c}_{i-1}$ appears within a row of $S(\mathbf{c}_i)$, and in particular, within a row of $S(\mathbf{c}_i)$ that directly shares an Alamouti or trivial 2×2 with $\mathbf{r}_s$.

Part 3) This proof is straight-forward; the details are left to the reader.

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