The Minimum Decoding Delay of Maximum Rate Complex Orthogonal Space–Time Block Codes

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***Abstract*—The growing demand for efficient wireless transmissions over fading channels motivated the development of space–time block codes. Space–time block codes built from generalized complex orthogonal designs are particularly attractive because the orthogonality permits a simple decoupled maximum-likelihood decoding algorithm while achieving full transmit diversity. The two main research problems for these complex orthogonal space–time block codes (COSTBCs) have been to determine for any number of antennas the maximum rate and the minimum decoding delay for a maximum rate code. The maximum rate for COSTBCs was determined by Liang in 2003. This paper addresses the second fundamental problem by providing a tight lower bound on the decoding delay for maximum rate codes. It is shown that for a maximum rate COSTBC for $2m-1$ or $2m$ antennas, a tight lower bound on decoding delay is $\tilde{r} = \binom{2m}{m-1}$. This lower bound on decoding delay is achievable when the number of antennas is congruent to 0, 1, or 3 modulo 4. This paper also derives a tight lower bound on the number of variables required to construct a maximum rate COSTBC for any given number of antennas. Furthermore, it is shown that if a maximum rate COSTBC has a decoding delay of r where $\tilde{r} < r \leq 2\tilde{r}$, then $r = 2\tilde{r}$. This is used to provide evidence that when the number of antennas is congruent to 2 modulo 4, the best achievable decoding delay is $2\binom{2m}{m-1}$.**

***Index Terms*—Generalized complex orthogonal design (GCOD), maximum rate, minimum decoding delay, multiple-input multiple-output (MIMO), space–time block code.**

I. INTRODUCTION

SPACE–TIME block codes have been widely studied for their applicability to multiple-input multiple-output (MIMO) wireless systems. Winters, Foschini, and Telatar each played significant roles in creating interest in MIMO systems [1]–[3], and Gesbert *et al.* have provided a detailed tutorial on MIMO space–time coded wireless systems [4]. Space–time block codes built from generalized complex orthogonal designs can be viewed as a generalization of Alamouti's scheme [5] and were introduced by Tarokh *et al.* [6]. These particular complex orthogonal space–time block codes (COSTBCs) are attractive because they can provide full transmit diversity while requiring a very simple decoupled maximum-likelihood decoding method [6], [7].

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A generalized complex orthogonal design (GCOD) $\boldsymbol{G}$ on k complex variables $z_1, \dots, z_k$, is an $r \times n$ matrix with entries $0, \pm z_1, \dots, \pm z_k, \pm z_1^*, \dots, \pm z_k^*$, satisfying

$$\boldsymbol{G}^H \boldsymbol{G} = \sum_{i=1}^{k} |z_i|^2 \boldsymbol{I}_n$$

where $\boldsymbol{G}^H$ denotes the Hermitian transpose of $\boldsymbol{G}$ and $\boldsymbol{I}_n$ is the $n \times n$ identity matrix. This definition requires that each column of $\boldsymbol{G}$ includes exactly one position occupied by z_i, $-z_i$, z_i^*, or $-z_i^*$, for each $i = 1, 2, \dots, k$, and each row has at most one position occupied by z_i, $-z_i$, z_i^*, or $-z_i^*$, for each $i = 1, 2, \dots, k$. Liang provides a review of variations on this definition [8].

An $r \times n$ GCOD on k variables can be utilized as a COSTBC wherein the n columns represent the transmissions of n transmit antennas, the k variables represent the k transmittable information symbols, and the number r of rows represents the decoding delay. The rate of the code is defined as $R = \frac{k}{r}$, the ratio of the number of information symbols to the decoding delay.

The two main research problems have been to determine the maximum rate for a GCOD with a given number of columns and to determine the minimum decoding delay (i.e., number of rows) for a maximum rate GCOD with a given number of columns. The first question was answered by Liang [7]; the second question, known as the "*fundamental question of generalized complex orthogonal design theory* [6]," is addressed in this paper.

Liang proved that to achieve full transmit diversity using a rectangular GCOD with $2m-1$ or $2m$ columns, the maximum achievable rate is $\frac{m+1}{2m}$ [7]. Furthermore, Liang provided an algorithm for constructing maximum rate GCODs for any number of columns [7]. Several other authors made progress towards determining the maximum rate and developed algorithms to produce high or maximum rate GCODs [9]–[11].

With the maximum rate question settled, attention focused on determining the minimum decoding delay of maximum rate GCODs. Until now, the minimum decoding delay had been addressed only in special cases. Liang proved the minimum decoding delay for GCODs with five and six columns [7], while Kan and Shen proved the minimum decoding delay for GCODs with seven columns and stated the minimum decoding delay for eight columns [12]. These proofs utilize arguments specialized to the specific number of columns involved.

Algorithms capable of generating GCODs with arbitrary numbers of columns [7], [13], recent breakthroughs in antenna technology [14], [15], and the growing interest in distributive systems have contributed to the interest in determining the minimum decoding delay for maximum rate GCODs with arbitrary numbers of columns. As the number of columns increases beyond four, classical mathematical results imply that there are no square complex orthogonal designs of maximum rate [6], [7], [16]–[18]. Therefore, when the number of columns is greater than four, the fundamental problem of determining the minimum decoding delay for maximum rate rectangular GCODs becomes increasingly important.

In this paper, we prove that for a maximum rate GCOD with $2m-1$ or $2m$ columns, a tight lower bound on the decoding delay is $\binom{2m}{m-1}$. This lower bound on decoding delay is achievable when the number of columns is congruent to 0, 1, or 3 modulo 4. When the number of columns is congruent to 2 modulo 4, we provide evidence that the best achievable decoding delay is $2\binom{2m}{m-1}$. This provides insight into which sizes of GCODs are most efficient in terms of rate and delay. We also determine a tight lower bound on the minimum number of variables required to build a maximum rate GCOD for any given number of columns.

In Section II, we present several results concerning maximum rate GCODs. In Section III, we provide a tight lower bound on the decoding delay of maximum rate GCODs. In Section IV, we provide a tight lower bound on the number of variables required in a maximum rate GCOD. We conclude our paper in Section V by discussing some implications of our results.

II. The Structure of Maximum Rate Designs

In this section, we review useful results and present new results concerning the structure of maximum rate GCODs. The proofs for the new results are contained in Appendix A.

Throughout this paper, we use the following *equivalence operations* which can be performed on any GCOD.

1) Rearrange the order in which the rows appear in the matrix ("row rearrangements").
2) Rearrange the order in which the columns appear in the matrix ("column rearrangements").
3) Conjugate and/or negate all instances of certain variables.
4) Multiply any row and/or column by -1.

For example, given a GCOD $\boldsymbol{G}$, we can perform row rearrangements to obtain a GCOD $\boldsymbol{G}'$ whose rows are the rows of $\boldsymbol{G}$ simply appearing in a different order. We say that $\boldsymbol{G}$ and $\boldsymbol{G}'$ are equivalent designs and that $\boldsymbol{G}'$ is simply a different *arrangement* of $\boldsymbol{G}$. Throughout this paper, we often consider different arrangements of a given design $\boldsymbol{G}$ by performing equivalence operations on the rows, columns, and/or entries of $\boldsymbol{G}$. Through a minor abuse of notation, we refer to any arrangement of a given GCOD $\boldsymbol{G}$ still as $\boldsymbol{G}$.

Throughout, we use Liang's result that for a GCOD with $2m-1$ or $2m$ columns, the maximum achievable rate is $\frac{m+1}{2m}$ [7]. Furthermore, we use Liang's result that for any variable z_i in a maximum rate GCOD $\boldsymbol{G}$, it is possible to arrange the matrix through suitable row and column rearrangements, suitable conjugations of all appearances of z_i, and suitable multiplications of rows or columns by -1, to obtain a submatrix $\boldsymbol{B}_i$ of the following form:

$$\boldsymbol{B}_i = \left(\begin{array}{cccc|cccc} z_i & 0 & \dots & 0 & & & & \\ 0 & z_i & \dots & 0 & & \boldsymbol{M}_i & & \\ \vdots & \vdots & \ddots & \vdots & & & & \\ 0 & 0 & \dots & z_i & & & & \\ \hline & & & & z_i^* & 0 & \dots & 0 \\ & -\boldsymbol{M}_i^H & & & 0 & z_i^* & \dots & 0 \\ & & & & \vdots & \vdots & \ddots & \vdots \\ & & & & 0 & 0 & \dots & z_i^* \end{array}\right).$$

Liang further shows that in the case of a maximum rate GCOD $\boldsymbol{G}$ with $2m-1$ columns, $\boldsymbol{B}_i$ is $2m-1 \times 2m-1$ and $\boldsymbol{M}_i$ is either $m \times m-1$ or $m-1 \times m$. In the case of a maximum rate GCOD with $2m$ columns, $\boldsymbol{B}_i$ is $2m \times 2m$ and $\boldsymbol{M}_i$ is $m \times m$ [7]. Throughout this paper, we assume that in the case where $\boldsymbol{G}$ has $2m-1$ columns, the submatrices $\boldsymbol{M}_i$ are size $m \times m-1$; all proofs can be altered slightly to hold if we instead allow the submatrices $\boldsymbol{M}_i$ to be of size $m-1 \times m$.

For easy reference, we recall the following result of Liang.

Result 2.1: [7] Let $\boldsymbol{G}$ be a maximum rate generalized complex orthogonal design on complex variables $z_1, z_2, \dots z_k$. Then, for any $1 \leq i \leq k$, given the submatrix $\boldsymbol{B}_i$ of $\boldsymbol{G}$, the portion $\boldsymbol{M}_i$ contains no zero entries.

For ease of discussion, we now present two definitions concerning the structure of maximum rate GCODs.

Definition 2.2: We say that a maximum rate generalized complex orthogonal design $\boldsymbol{G}$ is in "$\boldsymbol{B}_i$ form" if the submatrix $\boldsymbol{B}_i$ can be created in $\boldsymbol{G}$ through only row rearrangements, suitable conjugations and/or negations of all instances of z_i, and suitable multiplications of rows and/or columns by -1.

In other words, if $\boldsymbol{G}$ is in "$\boldsymbol{B}_i$ form," then every row of $\boldsymbol{B}_i$ appears within the rows of $\boldsymbol{G}$, up to possible conjugations of all instances of z_i and possible factors of -1. In the following sections, we will show that the importance of $\boldsymbol{G}$ being in "$\boldsymbol{B}_i$ form" is that the submatrix $\boldsymbol{B}_i$ can then be obtained within $\boldsymbol{G}$ through equivalence operations (namely, row rearrangements, conjugations, and multiplications by -1) that do not affect the zero patterns that appear within the rows of $\boldsymbol{G}$.

Definition 2.3: Consider a maximum rate generalized complex orthogonal design $\boldsymbol{G}$ with $2m-1$ or $2m$ columns. We define $\boldsymbol{G}_L$ to be the collection of the first m columns of any particular arrangement of $\boldsymbol{G}$ (that is, the left "half" of $\boldsymbol{G}$). We define $\boldsymbol{G}_R$ to be the collection of the last $m-1$ or m columns when the total number of columns is $2m-1$ or $2m$, respectively (that is, the right "half" of $\boldsymbol{G}$).

We now provide several results that will be used to derive our main theorems. The proofs of these results are contained in Appendix A. The first result, Lemma 2.4, contains a simple observation that is essential in the balance of this paper.

Lemma 2.4: Let $\boldsymbol{G}$ be a maximum rate generalized complex orthogonal design on variables $z_1, z_2, \ldots, z_k$. Then, for each $1 \leq i \leq k$, every instance of $\pm z_i$ or $\pm z_i^*$ in $\boldsymbol{G}$ must appear in the submatrix $\boldsymbol{B}_i$.

The following Lemma 2.5 explains that certain types of column rearrangements of a GCOD $\boldsymbol{G}$ in $\boldsymbol{B}_i$ form do not affect the $\boldsymbol{B}_i$ form.

Lemma 2.5: Let $\boldsymbol{G}$ be a maximum rate generalized complex orthogonal design with $2m-1$ or $2m$ columns on variables $z_1, z_2, \ldots, z_k$. If $\boldsymbol{G}$ is in $\boldsymbol{B}_i$ form for some $1 \leq i \leq k$, then $\boldsymbol{G}$ will remain in $\boldsymbol{B}_i$ form through any column rearrangements such that columns within $\boldsymbol{G}_L$ are rearranged exclusively within $\boldsymbol{G}_L$ and/or columns within $\boldsymbol{G}_R$ are rearranged exclusively within $\boldsymbol{G}_R$.

The following Lemma 2.6 can be used to show that a GCOD $\boldsymbol{G}$ is in $\boldsymbol{B}_i$ form if one row of $\boldsymbol{B}_i$ is known to appear within $\boldsymbol{G}$, up to conjugations and multiplications by -1.

Lemma 2.6: Let $\boldsymbol{G}$ be a maximum rate generalized complex orthogonal design with $2m-1$ or $2m$ columns on variables $z_1, z_2, \ldots, z_k$. If a single row $\boldsymbol{r}$ of $\boldsymbol{B}_i$, for some $1 \leq i \leq k$, is known to appear in $\boldsymbol{G}$ up to conjugation of the variable z_i and multiplications by -1, then $\boldsymbol{G}$ is in $\boldsymbol{B}_i$ form.

The following Corollary 2.7 is concerned with the effect of column swaps between $\boldsymbol{G}_L$ and $\boldsymbol{G}_R$ on a submatrix $\boldsymbol{B}_i$ of a design $\boldsymbol{G}$.

Corollary 2.7: Let $\boldsymbol{G}$ be a maximum rate generalized complex orthogonal design with $2m-1$ or $2m$ columns on complex variables $z_1, z_2, \ldots, z_k$. If $\boldsymbol{G}$ is in $\boldsymbol{B}_i$ form for some $1 \leq i \leq k$, then swapping columns between $\boldsymbol{G}_L$ and $\boldsymbol{G}_R$ either puts $\boldsymbol{G}$ in $\boldsymbol{B}_j$ form for some $j \neq i$ or puts $\boldsymbol{G}$ back into $\boldsymbol{B}_i$ form.

The final result in this section, Corollary 2.8, is critical for our proof of the main results in Sections III and IV.

Corollary 2.8: Let $\boldsymbol{G}$ be a maximum rate complex orthogonal design on variables $z_1, z_2, \ldots, z_k$. Then any arrangement of $\boldsymbol{G}$ is in $\boldsymbol{B}_i$ form for some $1 \leq i \leq k$.

III. Minimum Decoding Delay for Maximum Rate Designs

We are now prepared to provide a tight lower bound on the decoding delay of maximum rate GCODs. We derive the lower bound using the results presented in Section II, and then we show that the bound is tight by referring to an algorithm presented by Lu, Fu, and Xia [13].

Theorem 3.1: A tight lower bound on the decoding delay of a maximum rate generalized complex orthogonal design $\boldsymbol{G}$ with $2m-1$ or $2m$ columns is

$$\tilde{r} = \binom{2m}{m-1}.$$

If the number of columns $2m-1$ or $2m$ is congruent to 0, 1, or 3 modulo 4, then this lower bound on decoding delay is achievable.

Proof: We first prove the case where $\boldsymbol{G}$ has $2m$ columns. Let $z_1, z_2, \ldots, z_k$ be the variables in $\boldsymbol{G}$. By Corollary 2.8, any arrangement of the columns of $\boldsymbol{G}$ is in $\boldsymbol{B}_i$ form for some $1 \leq i \leq k$. Now, consider any $m-1$ columns in $\boldsymbol{G}_L$ for a given arrangement of $\boldsymbol{G}$. Since $\boldsymbol{G}$ is in some $\boldsymbol{B}_i$ form, the structure of $\boldsymbol{B}_i$ and Result 2.1 together imply that there is a row in $\boldsymbol{G}$ that has zero entries in each of the $m-1$ chosen columns of $\boldsymbol{G}_L$, z_i (up to conjugation and sign) in the remaining column of $\boldsymbol{G}_L$, and nonzero entries in each of the m columns of $\boldsymbol{G}_R$.

More generally, given any $m-1$ columns of $\boldsymbol{G}$, there must be at least one row in $\boldsymbol{G}$ that has zero entries in these $m-1$ columns and nonzero entries in each of the remaining columns. To see this, notice that given any choice of $m-1$ columns of $\boldsymbol{G}$, these $m-1$ columns can be placed together in $\boldsymbol{G}_L$. Then, by Corollary 2.8, this new arrangement of $\boldsymbol{G}$ will be in $\boldsymbol{B}_j$ form for some $1 \leq j \leq k$. Hence, by the structure of $\boldsymbol{B}_j$ and Result 2.1, there must be a row in $\boldsymbol{G}$ that has zero entries in each of the $m-1$ chosen columns of $\boldsymbol{G}_L$, z_j (up to conjugation and sign) in the remaining column of $\boldsymbol{G}_L$, and nonzero entries in each of the m columns of $\boldsymbol{G}_R$.

Thus, we have shown that every possible pattern of exactly $m-1$ zeros in a row of length $2m$ must appear within a distinct row of $\boldsymbol{G}$. Therefore, we may conclude that for a maximum rate $\boldsymbol{G}$ with $2m$ columns, the minimum number of rows required (i.e., the minimum decoding delay) is bounded below by $\tilde{r} = \binom{2m}{m-1}$.

We now prove the case where $\boldsymbol{G}$ has $2m-1$ columns. By Corollary 2.8, we may assume that $\boldsymbol{G}$ is in $\boldsymbol{B}_i$ form for some $1 \leq i \leq k$. Then, for any $m-1$ chosen columns of $\boldsymbol{G}_L$, there is a row in $\boldsymbol{B}_i$ that has zero entries in each of the $m-1$ chosen columns of $\boldsymbol{G}_L$, z_i (up to conjugation and sign) in the remaining column of $\boldsymbol{G}_L$, and nonzero entries in each of the $m-1$ columns of $\boldsymbol{G}_R$. Similarly, for any $m-2$ chosen columns of $\boldsymbol{G}_R$, there is a row in $\boldsymbol{B}_i$ that has zero entries in each of the $m-2$ chosen columns of $\boldsymbol{G}_R$, z_i^* (up to conjugation and sign) in the remaining column of $\boldsymbol{G}_R$, and nonzero entries in each of the m columns of $\boldsymbol{G}_L$. Moreover, since any arrangement of the columns of $\boldsymbol{G}$ results in $\boldsymbol{B}_j$ form for some z_j, it follows that 1) for any $m-1$ columns of $\boldsymbol{G}$, there must be at least one row of $\boldsymbol{G}$ that has zero entries in each of these $m-1$ columns and nonzero entries in each of the remaining columns; and 2) for any $m-2$ columns of $\boldsymbol{G}$, there must be at least one row of $\boldsymbol{G}$ that has zero entries in each of these $m-2$ columns and nonzero entries in each of the remaining columns. Hence, every possible pattern of exactly $m-1$ zeros in a row of length $2m-1$ must appear within a distinct row of $\boldsymbol{G}$, and every possible pattern of exactly $m-2$ zeros in a row of length $2m-1$ must appear within a distinct row of $\boldsymbol{G}$. We may conclude that for a maximum rate $\boldsymbol{G}$ with $2m-1$ columns, the minimum decoding delay is bounded below by

$$\tilde{r} = \binom{2m-1}{m-1} + \binom{2m-1}{m-2}.$$

By Pascal's Identity, we can conclude that $\tilde{r} = \binom{2m}{m-1}$.

We conclude that when $\boldsymbol{G}$ has either $2m-1$ or $2m$ columns, the decoding delay is bounded below by $\tilde{r} = \binom{2m}{m-1}$.

When the number of columns is congruent to 0, 1, or 3 modulo 4, this means, respectively, that we have $2m$ columns where m is even, $2m-1$ columns where m is odd, or $2m-1$

columns where m is even. To show that in these cases we can achieve the lower bound on decoding delay, we refer to the work completed by Lu, Fu, and Xia [13]. These authors found a closed-form description for designs with $2m-1$ or $2m$ columns with rate $\frac{m+1}{2m}$, which is now known to be the maximum achievable rate, and they provided a formula for the decoding delay of their designs. For the case of $2m$ columns, where m is even (described equivalently as the case for $n+3$ antennas where $n = 2m-1$ and m is odd), they give a construction proven to have a decoding delay which simplifies to $\binom{2m}{m-1}$. In the case of $2m-1$ columns, their proven delay also simplifies to $\binom{2m}{m-1}$. Therefore, since we have proven that the lower bound for decoding delay for $2m-1$ or $2m$ columns is $\binom{2m}{m-1}$, and since Lu, Fu, and Xia have demonstrated construction methods proven to obtain this delay when the total number of antennas is congruent to 0, 1, or 3 modulo 4, we can conclude that for these cases, we can achieve the minimum possible decoding delay. □

Theorem 3.1 completely answers the question as to the minimum achievable decoding delay for maximum rate GCODs with the number of columns equivalent to 0, 1, or 3 modulo 4. Our next result, Theorem 3.2, will be used to provide information concerning the minimum achievable decoding delay for the case where the number of columns is equivalent to 2 modulo 4.

Theorem 3.2: Let $\boldsymbol{G}$ be a maximum rate complex orthogonal design $\boldsymbol{G}$ with $2m-1$ or $2m$ columns. If two distinct rows of $\boldsymbol{G}$ have the same zero pattern, then every permissible zero pattern must appear in at least two distinct rows and the minimum achievable decoding delay is $2\binom{2m}{m-1}$, twice the lower bound on delay.

Proof: Let $z_1, z_2, \ldots, z_k$ be the variables in $\boldsymbol{G}$. Suppose that $\boldsymbol{G}$ contains two rows $\boldsymbol{r}_a$ and $\boldsymbol{r}_b$ that have identical zero patterns, i.e., the placement of the zero entries within $\boldsymbol{r}_a$ and $\boldsymbol{r}_b$ are identical.

Suppose for contradiction that $\boldsymbol{r}_a$ and $\boldsymbol{r}_b$ both contain an entry from $\{\pm z_i, \pm z_i^*\}$ for some $1 \le i \le k$. Then, by Lemma 2.4, there is some series of equivalence operations that can be performed on $\boldsymbol{G}$ that transform row $\boldsymbol{r}_a$ into a row $\boldsymbol{r}'_a$ in $\boldsymbol{B}_i$, up to conjugations and multiplications by -1; similarly for $\boldsymbol{r}_b$. Since all rows of $\boldsymbol{B}_i$, up to conjugations and multiplications by -1, must simultaneously appear in $\boldsymbol{G}$ when $\boldsymbol{G}$ is in $\boldsymbol{B}_i$ form, there must be some set of column rearrangements, conjugations, and possible multiplications by -1 that simultaneously transform $\boldsymbol{r}_a$ into a row of $\boldsymbol{B}_i$ and transform $\boldsymbol{r}_b$ into a row of $\boldsymbol{B}_i$. Since $\boldsymbol{r}_a$ and $\boldsymbol{r}_b$ began with identical zero patterns, they must be transformed into rows $\boldsymbol{r}'_a$ and $\boldsymbol{r}'_b$ with identical zero patterns. However, by the structure of $\boldsymbol{B}_i$, there are no rows in $\boldsymbol{B}_i$ with identical zero patterns. Hence, if $\boldsymbol{r}_a$ and $\boldsymbol{r}_b$ have identical zero patterns, then these rows cannot contain any common variables.

It follows from Lemma 2.4 that every row (or some permutation thereof) in $\boldsymbol{G}$ appears in a submatrix $\boldsymbol{B}_\ell$, for some $1 \le \ell \le k$, up to conjugations and multiplications by -1. Suppose then that some permutation of $\boldsymbol{r}_a$ (up to conjugations and signs) appears within $\boldsymbol{B}_a$ for some variable z_a. Similarly, suppose that some permutation of $\boldsymbol{r}_b$ (up to conjugations and signs) appears within $\boldsymbol{B}_b$ for some variable z_b. Since $\boldsymbol{r}_a$ and $\boldsymbol{r}_b$ can contain no common variables, we must assume that $z_a \neq z_b$. Then, since $\boldsymbol{r}_a$ and $\boldsymbol{r}_b$ have identical zero patterns, and by Result 2.1, we can rearrange the columns of $\boldsymbol{G}$ so that rows $\boldsymbol{r}_a$ and $\boldsymbol{r}_b$ are transformed into rows $\boldsymbol{r}''_a$ and $\boldsymbol{r}''_b$ with all nonzero elements in the m columns of $\boldsymbol{G}_L$, nonzero elements z_a and z_b (up to conjugation and sign), respectively, in the $(m+1)$th column, followed by all zeros in the remaining columns of $\boldsymbol{G}_R$. Clearly, $\boldsymbol{r}''_a$ is a row of $\boldsymbol{B}_a$, and $\boldsymbol{r}''_b$ is a row of $\boldsymbol{B}_b$ (up to conjugations and signs). By Corollary 2.6, if a single row of $\boldsymbol{B}_a$ appears in $\boldsymbol{G}$, up to conjugations and signs, then all rows of $\boldsymbol{B}_a$ appear in $\boldsymbol{G}$, up to conjugations and signs. Similarly, all rows of $\boldsymbol{B}_b$ must appear in $\boldsymbol{G}$, up to conjugations and signs. We note that conjugations and multiplications by -1 do not affect the zero patterns of the rows in $\boldsymbol{B}_a$ or $\boldsymbol{B}_b$.

We now explain that it is not possible for a single row to simultaneously be, even up to conjugations and multiplications by -1, in $\boldsymbol{B}_a$ and $\boldsymbol{B}_b$ for $a \neq b$. A row $\boldsymbol{r}$ (up to conjugations and multiplication by -1) in $\boldsymbol{B}_a$ either has its only nonzero element in $\boldsymbol{G}_L$ as z_a or its only nonzero element in $\boldsymbol{G}_R$ as z_a^*, and similarly for a row in $\boldsymbol{B}_b$. Since $a \neq b$, a single row $\boldsymbol{r}$ cannot simultaneously be contained in both $\boldsymbol{B}_a$ and $\boldsymbol{B}_b$.

So, given two rows in $\boldsymbol{G}$ with the same zero pattern, there are rearrangements of columns, conjugations of all appearances of certain variables, and possible multiplications by -1 that show $\boldsymbol{G}$ to contain two distinct submatrices $\boldsymbol{B}_a$ and $\boldsymbol{B}_b$, with no overlap of rows between submatrices. We now proceed with an argument similar to the one utilized in the proof of Theorem 3.1 by considering the arrangement of $\boldsymbol{G}$ containing the submatrices $\boldsymbol{B}_a$ and $\boldsymbol{B}_b$. First, in the case of $2m$ columns, choose any $m-1$ columns from $\boldsymbol{G}_L$. Then, due to the structure of $\boldsymbol{B}_a$ and $\boldsymbol{B}_b$ and Result 2.1, there will be two rows in $\boldsymbol{G}$ that contain zero entries in each of the chosen $m-1$ columns and nonzero entries in each of the remaining columns. Similarly, given any choice of $m-1$ columns in $\boldsymbol{G}$, these columns can be put into $\boldsymbol{G}_L$ and, by Corollary 2.8, this new arrangement of $\boldsymbol{G}$ will contain the distinct rows of $\boldsymbol{B}_c$ and $\boldsymbol{B}_d$, for some z_c and z_d (up to conjugation and sign). So, there must be two rows within $\boldsymbol{G}$ that contain zero in each of the chosen $m-1$ columns and nonzero entries in each of the remaining columns. It follows then that every possible pattern of $m-1$ zeros must appear at least twice within the length $2m$ rows of $\boldsymbol{G}$. So, if one zero pattern appears in two rows of a maximum rate GCOD, then the minimum achievable decoding delay is $2\binom{2m}{m-1}$. Now, in the case with $2m-1$ columns, a similar argument shows that every possible pattern of exactly $m-1$ zeros and every possible pattern of exactly $m-2$ zeros must appear twice within the length $2m-1$ rows of $\boldsymbol{G}$. Hence, in this case, the minimum achievable decoding delay is

$$2\binom{2m-1}{m-1} + 2\binom{2m-1}{m-2} = 2\binom{2m}{m-1}.$$

We conclude that if one allowable zero pattern appears in two rows of a maximum rate GCOD $\boldsymbol{G}$ with $2m-1$ or $2m$ columns, then the minimum achievable decoding delay is $2\binom{2m}{m-1}$, which is twice the lower bound on minimum decoding delay. Since a maximum rate GCOD that achieves the lower bound on decoding delay contains each allowable zero pattern in exactly one row, it follows directly that if a maximum rate GCOD has a decoding delay r with $\tilde{r} < r \le 2\tilde{r}$, then $r = 2\tilde{r}$. □

To illustrate Theorem 3.2, we note that each example with $2m$ columns produced by Liang's [7] well-known algorithm has a decoding delay of exactly twice the lower bound on delay, and each such example has every allowable pattern of zeros appearing in exactly two rows. In certain cases, Liang's algorithm can be amended to produce examples that do achieve the lower bound on minimum decoding delay. For example, we can uniquely extend Liang's example with seven columns to produce an example with eight columns that does achieve the lower bound. In (1) in Appendix B, we provide the design with eight columns that we obtained by extending Liang's example with seven columns [7]. Indeed, (1) is a design with $m = 4$ and $\binom{2(4)}{4-1} = \binom{8}{3} = 56$ rows, achieving the lower bound on decoding delay. The authors note that during revisions of this manuscript, they noticed that Liang independently demonstrated this same extension from seven columns to eight columns [19], but it was not known at that time that this extension produced a code with the minimum achievable decoding delay.

Theorem 3.2 can also be used to provide evidence of the following conjecture.

Conjecture 3.3: Let $\boldsymbol{G}$ be a maximum rate generalized complex orthogonal design. If the number of columns $2m$ is congruent to 2 modulo 4, then the decoding delay of $\boldsymbol{G}$ can be at best $2\binom{2m}{m-1}$, twice the lower bound.

Partial Proof: When the number of columns in a maximum rate GCOD is congruent to 2 modulo 4, experimental evidence suggests that it is not possible for the decoding delay to achieve the lower bound of Theorem 3.1. The issue that seems to prevent us from achieving the lower bound on decoding delay stems from the required distribution of negative signs. If it is correct that we cannot achieve the lower bound on decoding delay in this case, then our Theorem 3.2 implies that the minimum achievable decoding delay could be at best $2\binom{2m}{m-1}$, twice the lower bound. Lu, Fu, and Xia have demonstrated a construction algorithm proven to achieve a decoding delay of $2\binom{2m}{m-1}$ in this case [13]. Additional algorithms [7], [11] also produce codes that achieve this same delay. Therefore, we expect that the minimum achievable decoding delay in this case is twice the lower bound on minimum decoding delay. △

During revisions of this manuscript, we found that an equivalent conjecture has been made without supporting discussion by Kan and Shen [12].

IV. Minimum Number of Variables in Maximum Rate Designs

In this section, we derive a formula for a tight lower bound on the number of variables required to build a maximum rate GCOD.

Corollary 4.1: Suppose that $\boldsymbol{G}$ is a maximum rate complex orthogonal design with $2m-1$ or $2m$ columns. Then a tight lower bound on the number of variables required is $\frac{1}{2}\binom{2m}{m}$. This lower bound can be achieved when the number of columns is equivalent to 0, 1, or 3 modulo 4.

Proof: Recall that a maximum rate GCOD with $2m-1$ or $2m$ columns has rate $\frac{k}{r} = \frac{m+1}{2m}$ [7], where k represents the total number of distinct complex variables in the GCOD and r represents the decoding delay. Then, the lower bound $\tilde{r}$ provided in Theorem 3.1 gives the following:

$$\begin{aligned} k &= r\frac{m+1}{2m} \\ &\geq \tilde{r}\frac{m+1}{2m} \\ &= \frac{1}{2}\binom{2m}{m}. \end{aligned}$$

Hence, the minimum number of variables in a maximum rate complex orthogonal design with $2m-1$ or $2m$ columns is $\frac{1}{2}\binom{2m}{m}$. The construction provided by Lu, Fu, and Xia [13] shows that this bound is tight in the cases where the number of columns is congruent to $0, 1$, or 3 modulo 4. □

We note that the eight-column GCOD in (1) obtained by extending Liang's example with seven columns has $\frac{1}{2}\binom{2(4)}{4} = 35$ variables, achieving the minimum number of variables necessary to produce a maximum rate GCOD with eight columns.

V. Conclusion

This paper exploits newly discovered combinatorial properties of maximum rate GCODs to address the "*fundamental question of generalized complex orthogonal design theory* [6]" by determining a tight lower bound on the decoding delay of maximum rate GCODs.

The results concerning the minimum decoding delay can be summarized as follows: 1) A lower bound on decoding delay for maximum rate GCODs with $2m-1$ or $2m$ columns is $\tilde{r} = \binom{2m}{m-1}$. 2) This lower bound $\tilde{r}$ is achievable when the number of columns is congruent to 0, 1, or 3 modulo 4 [13]. 3) If a maximum rate GCOD has a decoding delay r with $\tilde{r} < r \leq 2\tilde{r}$, then $r = 2\tilde{r}$. Hence, if a maximum rate GCOD does not achieve the lower bound on delay, then it can achieve at best twice the lower bound on decoding delay. 4) Twice the lower bound on decoding delay is achievable when the number of columns is congruent to 2 modulo 4 [13].

For an alternative view of the structural constraints of maximum rate GCODs, we proved that a tight lower bound on the number of variables required to build a maximum rate GCOD with $2m-1$ or $2m$ columns is $\frac{1}{2}\binom{2m}{m}$. This lower bound is achievable when the number of columns is congruent to $0, 1$, or 3 modulo 4 [13].

The minimum decoding delay provides an evaluation criterion for comparing maximum rate COSTBCs: We want to chose those maximum rate codes that achieve the minimum decoding delay. Given that codes for $2m-1$ and $2m$ transmit antennas achieve the same maximum rate of $\frac{m+1}{2m}$, we conjecture that transmitting with a code designed for $2m$ antennas where m is odd is undesirable. It appears to be preferable to use one fewer antenna $(2m-1)$ and cut the delay in half. In future applications requiring many antennas, we recommend using $2m$ antennas where m is even.

By determining a tight lower bound on the decoding delay for arbitrary maximum rate COSTBCs, we have shown that the minimum decoding delay grows quickly with respect to the number of antennas. The maximum rate approaches $1/2$

as the number of antennas increases, and for large numbers of columns, the reduction of rate from $\frac{m+1}{2m}$ to $1/2$ may be insignificant compared to the potential reduction in decoding delay. Therefore, as interest grows in applications involving large numbers of antennas, it will become increasingly important to study the minimum achievable decoding delay for rate $1/2$ COSTBCs.

APPENDIX A

This appendix contains the proofs of the results presented in Section II.

Proof of Lemma 2.4: In our GCODs, each variable or its conjugate appears exactly once per column. Also, for each $1 \leq i \leq k$, we can arrange $\boldsymbol{G}$ using equivalence operations so that $\boldsymbol{G}$ will contain all rows of the submatrix $\boldsymbol{B}_i$, up to conjugation of all instances of z_i and multiplications by -1 [7]. Hence, the instances of $\pm z_i$ or $\pm z_i^*$ appearing in each column of $\boldsymbol{B}_i$, up to conjugation and multiplications by -1, must be exactly the total instances of $\pm z_i$ or $\pm z_i^*$ in $\boldsymbol{G}$. □

Proof of Lemma 2.5: Suppose that $\boldsymbol{G}$ is in $\boldsymbol{B}_i$ form for some $1 \leq i \leq k$. Then, up to conjugations and multiplications by -1, the $2m-1$ or $2m$ rows (according to if $\boldsymbol{G}$ has $2m-1$ or $2m$ columns, respectively) of $\boldsymbol{B}_i$ appear in some order in $\boldsymbol{G}$. We may rearrange the rows such that the $2m-1$ (or $2m$) rows of $\boldsymbol{B}_i$ (up to conjugations and multiplications by -1) are the first $2m-1$ (or $2m$) rows in $\boldsymbol{G}$. Label these rows $\boldsymbol{r}_1, \ldots, \boldsymbol{r}_{2m-1}, (\boldsymbol{r}_{2m})$. Then, suppose a permutation σ maps the columns $\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_m$ in $\boldsymbol{G}_L$ to $\sigma(\boldsymbol{c}_1), \sigma(\boldsymbol{c}_2), \ldots, \sigma(\boldsymbol{c}_m)$ in $\boldsymbol{G}_L$. Then, we can apply the same permutation σ to rows $\boldsymbol{r}_1, \ldots, \boldsymbol{r}_m$, which will restore $\boldsymbol{B}_i$ (up to conjugations and multiplications by -1) in the top portion of $\boldsymbol{G}$. Similarly, after a permutation σ permuting the $m-1$ (or m) columns of $\boldsymbol{G}_R$ within $\boldsymbol{G}_R$, we can apply σ to the $m-1$ (or m) rows $\boldsymbol{r}_{m+1}, \ldots, \boldsymbol{r}_{2m-1}, (\boldsymbol{r}_{2m})$ that correspond to the last $m-1$ (or m) rows of $\boldsymbol{B}_i$ to restore $\boldsymbol{B}_i$ (up to conjugations and multiplications by -1) in the top portion of $\boldsymbol{G}$. Hence, $\boldsymbol{G}$ remains in $\boldsymbol{B}_i$ form after rearranging columns exclusively within $\boldsymbol{G}_L$ and/or exclusively within $\boldsymbol{G}_R$. □

Proof of Lemma 2.6: We prove here the case with $2m$ columns, as the case with $2m-1$ columns is similar. First, note that given $\boldsymbol{G}$, for each $1 \leq i \leq k$, there are suitable column and row rearrangements, suitable conjugations and/or negations of all instances of certain variables, and suitable multiplications of rows and columns by -1 that produce the submatrix $\boldsymbol{B}_i$ within the design $\boldsymbol{G}$ [7]. So, by Definition 2.2, it suffices to determine which, if any, rearrangements of columns of $\boldsymbol{G}$ are necessary to achieve $\boldsymbol{B}_i$ form.

We assume that some row $\boldsymbol{r}$ of $\boldsymbol{B}_i$, for some $1 \leq i \leq k$, is known to appear, up to conjugations and multiplications by -1, in a maximum rate GCOD $\boldsymbol{G}$. Now, suppose for contradiction that $\boldsymbol{G}$ is not in $\boldsymbol{B}_i$ form. Then, we must be able to achieve $\boldsymbol{B}_i$ form by rearranging the columns of $\boldsymbol{G}$ in some way.

First, we consider rearranging the columns of $\boldsymbol{G}$ such that columns within $\boldsymbol{G}_L$ are rearranged exclusively within $\boldsymbol{G}_L$ and columns within $\boldsymbol{G}_R$ are rearranged exclusively within $\boldsymbol{G}_R$. If we can achieve $\boldsymbol{B}_i$ form through column rearrangements of this type, then Lemma 2.5 implies that $\boldsymbol{G}$ was already in $\boldsymbol{B}_i$ form. However, this contradicts our assumption that $\boldsymbol{G}$ was not initially in $\boldsymbol{B}_i$ form.

So, we must consider the case where column rearrangements involve swapping a column from $\boldsymbol{G}_L$ with a column from $\boldsymbol{G}_R$. We will consider the impact of such column rearrangements on $\boldsymbol{r}$, the row in the initial arrangement of $\boldsymbol{G}$ that is known to be a row of $\boldsymbol{B}_i$, up to conjugation of z_i and multiplication by -1. Note that, up to conjugation of z_i and multiplications by -1, either $\boldsymbol{r}$ belongs to the first m rows of $\boldsymbol{B}_i$, or $\boldsymbol{r}$ belongs to the last m rows of $\boldsymbol{B}_i$. It follows from the structure of $\boldsymbol{B}_i$ and Result 2.1 that in the former case, $\boldsymbol{r}$ contains one instance of $\pm z_i$ (or $\pm z_i^*$) and $m-1$ zeros within $\boldsymbol{G}_L$ and contains m nonzero entries within $\boldsymbol{G}_R$. In the latter case, $\boldsymbol{r}$ contains within $\boldsymbol{G}_L$ all nonzero entries and contains within $\boldsymbol{G}_R$ one instance of $\pm z_i^*$ (or $\pm z_i$) and $m-1$ zeros. We assume the former case; the latter follows similarly.

We consider the effect on $\boldsymbol{r}$ of two types of swaps between $\boldsymbol{G}_L$ and $\boldsymbol{G}_R$. In a Type 1 swap, a zero entry in $\boldsymbol{r}$ from $\boldsymbol{G}_L$ is exchanged with a nonzero entry in $\boldsymbol{r}$ in $\boldsymbol{G}_R$. Then the row $\boldsymbol{r}'$ obtained from $\boldsymbol{r}$ after this swap will have two nonzero entries in $\boldsymbol{G}_L$ and only $m-1$ nonzero entries in $\boldsymbol{G}_R$. By the definition of $\boldsymbol{B}_i$, this row $\boldsymbol{r}'$ containing $\pm z_i$ (or $\pm z_i^*$) is no longer in $\boldsymbol{B}_i$ (even up to conjugations or multiplications by -1). Hence, by Proposition 2.4, $\boldsymbol{G}$ is not in $\boldsymbol{B}_i$ form. Therefore, we cannot achieve $\boldsymbol{B}_i$ form by performing a single Type 1 swap between columns in $\boldsymbol{G}_L$ and $\boldsymbol{G}_R$.

In a Type 2 swap, the column swap exchanges the main diagonal entry $\pm z_i$ (or $\pm z_i^*$) of $\boldsymbol{r}$ within $\boldsymbol{G}_L$ with some other nonzero variable $\pm z_j$ or $\pm z_j^*$ appearing in row $\boldsymbol{r}$ in $\boldsymbol{G}_R$. Then the row $\boldsymbol{r}''$ obtained from $\boldsymbol{r}$ after this swap has $\pm z_j$ or $\pm z_j^*$ and $m-1$ zeros appearing in $\boldsymbol{G}_L$ and m nonzero entries (including $\pm z_i$ or $\pm z_i^*$) appearing in $\boldsymbol{G}_R$. By definition of $\boldsymbol{B}_i$, this row $\boldsymbol{r}''$ containing $\pm z_i$ (or $\pm z_i^*$) no longer appears in $\boldsymbol{B}_i$ (even up to conjugations or multiplications by -1). Hence, by Proposition 2.4, $\boldsymbol{G}$ is not in $\boldsymbol{B}_i$ form. (In fact, $\boldsymbol{r}''$ is now a row in $\boldsymbol{B}_j$, up to conjugations and multiplications by -1.) Therefore, we cannot achieve $\boldsymbol{B}_i$ form by performing a single Type 2 swap between columns in $\boldsymbol{G}_L$ and $\boldsymbol{G}_R$.

So, we have shown that any single swap of columns between $\boldsymbol{G}_L$ and $\boldsymbol{G}_R$ converts the row $\boldsymbol{r}$ of $\boldsymbol{B}_i$ (up to conjugations and multiplications by -1) into a row that no longer belongs to $\boldsymbol{B}_i$ (even up to conjugations and multiplication by -1). This implies that single column swaps of this type cannot be used to convert $\boldsymbol{G}$ into $\boldsymbol{B}_i$ form. Any nontrivial series of such swaps will also convert the row $\boldsymbol{r}$ of $\boldsymbol{B}_i$ (up to conjugations and multiplications by -1) into a row that no longer belongs to $\boldsymbol{B}_i$ (even up to conjugations and multiplications by -1). However, we must consider the trivial special case wherein each of the m columns in $\boldsymbol{G}_L$ is swapped with a different one of the m columns in $\boldsymbol{G}_R$. In this special case, the row $\boldsymbol{r}$ from $\boldsymbol{B}_i$ (up to conjugations and multiplications by -1) is converted into a row that is again a row of $\boldsymbol{B}_i$ (up to conjugations and multiplications by -1). However, if we are able to achieve $\boldsymbol{B}_i$ form through this special series of column swaps, then we must have already been in $\boldsymbol{B}_i$ form before this special series of column swaps. This follows since this special series of column swaps serves only to exchange the role of the submatrix $z_i\boldsymbol{I}$ with $z_i^*\boldsymbol{I}$ and the role of the submatrix $\boldsymbol{M}_i$ with $-\boldsymbol{M}_i^H$.

It follows that if $\boldsymbol{G}$ contains one row $\boldsymbol{r}$ of $\boldsymbol{B}_i$, up to conjugations and multiplications by -1, then any series of column rearrangements either precludes the possibility that $\boldsymbol{G}$ has achieved $\boldsymbol{B}_i$ form, or achieves $\boldsymbol{B}_i$ form while implying that $\boldsymbol{G}$ must have also initially been in $\boldsymbol{B}_i$ form. But, since $\boldsymbol{B}_i$ form must be achievable, we have shown in all cases that if $\boldsymbol{G}$ contains one row $\boldsymbol{r}$ of $\boldsymbol{B}_i$, up to conjugation and multiplication by -1, then $\boldsymbol{G}$ is already in $\boldsymbol{B}_i$ form. □

Proof of Corollary 2.7: We prove here the case with $2m$ columns, as the case with $2m-1$ columns is entirely similar. Suppose that $\boldsymbol{G}$ is in $\boldsymbol{B}_i$ form, so that $\boldsymbol{G}$ clearly contains a row $\boldsymbol{r}$ of $\boldsymbol{B}_i$, up to conjugations and multiplications by -1. Then, by the proof of Lemma 2.6, any series of column swaps between groups $\boldsymbol{G}_L$ and $\boldsymbol{G}_R$ either eliminates the current $\boldsymbol{B}_i$ form or, in the trivial special case where the left and right sides of $\boldsymbol{G}$ are completely swapped, restores $\boldsymbol{B}_i$ form. We now show that in the former case, $\boldsymbol{G}$ is moved into $\boldsymbol{B}_j$ form for some $j \neq i$.

The proof of Lemma 2.6 shows that a Type 1 swap σ of a column in $\boldsymbol{G}_L$ with a column in $\boldsymbol{G}_R$ will change row $\boldsymbol{r}$ in $\boldsymbol{B}_i$ into a row $\sigma(\boldsymbol{r}) = \boldsymbol{r}'$, which is a row with two nonzero entries in $\boldsymbol{G}_L$ and $m-1$ nonzero entries in $\boldsymbol{G}_R$. Specifically, suppose that $\boldsymbol{r}$ is the rth row of $\boldsymbol{G}$ and it contains z_i (up to conjugation and sign) in entry (r,t), where t is a column within $\boldsymbol{G}_L$. Suppose that σ exchanges column $a \neq t$ of $\boldsymbol{G}_L$ with column b of $\boldsymbol{G}_R$, whose entry (r,b) is z_ℓ (for some $1 \leq \ell \neq i \leq k$, up to conjugation and sign). In short, σ exchanges the entry (r,a) in $\boldsymbol{G}_L$ with the entry (r,b) in $\boldsymbol{G}_R$. Then, because $\boldsymbol{G}$ started in $\boldsymbol{B}_i$ form prior to this swap, there was another row, say $\boldsymbol{s}$, the sth row in $\boldsymbol{G}$, that had the variable z_i^* (up to conjugation and sign) in entry (s,b) within $\boldsymbol{G}_R$, all other entries as 0 within $\boldsymbol{G}_R$, and all nonzero elements within $\boldsymbol{G}_L$. In particular, say entry (s,a) within $\boldsymbol{G}_L$ is z_j (up to conjugation and sign). Hence, $\sigma(\boldsymbol{s}) = \boldsymbol{s}'$ is a row with all zeros in $\boldsymbol{G}_R$ except for z_j (up to conjugation and sign) in entry (s,b) and all nonzeros in $\boldsymbol{G}_L$. (In fact, performing a Type 1 swap with respect to row $\boldsymbol{r}$ is equivalent to performing a Type 2 swap with respect to row $\boldsymbol{s}$.) Hence, $\boldsymbol{s}'$ is a row of $\boldsymbol{B}_j$, up to conjugation of all appearances of z_j and possible multiplication by -1. Then, by Lemma 2.6, $\boldsymbol{G}$ is in $\boldsymbol{B}_j$ form. It follows directly that any nontrivial series of Type 1 swaps will also move $\boldsymbol{G}$ into some $\boldsymbol{B}_j$ form.

The proof of Lemma 2.6 shows that a Type 2 swap will change row $\boldsymbol{r}$ of $\boldsymbol{B}_i$, up to conjugation and multiplication by -1, into a row $\boldsymbol{r}''$ which is a row of $\boldsymbol{B}_j$, up to conjugations and multiplications by -1. By Lemma 2.6, since $\boldsymbol{r}''$ is a row of $\boldsymbol{B}_j$ up to conjugations and multiplications by -1, $\boldsymbol{G}$ is in $\boldsymbol{B}_j$ form. It follows directly that any nontrivial series of Type 2 swaps will also move $\boldsymbol{G}$ into some $\boldsymbol{B}_j$ form.

Therefore, we can conclude that given a design $\boldsymbol{G}$ in $\boldsymbol{B}_i$ form, any series of column swaps between $\boldsymbol{G}_L$ and $\boldsymbol{G}_R$ results in the creation of a $\boldsymbol{B}_j$ form, either through the trivial special case where we regain $\boldsymbol{B}_i$ form, or where the $\boldsymbol{B}_i$ form is eliminated and a new $\boldsymbol{B}_j$ form is created with $j \neq i$. □

Proof of Corollary 2.8: Consider an arbitrary arrangement of the rows, columns, and entries of a GCOD $\boldsymbol{G}$, and call this the *initial arrangement* of $\boldsymbol{G}$. Since the submatrix $\boldsymbol{B}_1$ must be achievable [7], Definition 2.2 implies that we can perform appropriate column rearrangements to put $\boldsymbol{G}$ in $\boldsymbol{B}_1$ form. Now, we

$$
\left(\begin{array}{cccccccc}
z_1 & 0 & 0 & 0 & z_2 & z_3 & z_4 & z_5 \\
0 & z_1 & 0 & 0 & z_6 & z_7 & z_8 & z_9 \\
0 & 0 & z_1 & 0 & z_{10} & z_{11} & z_{12} & z_{13} \\
0 & 0 & 0 & z_1 & -z_{35}^* & z_{34}^* & -z_{33}^* & z_{32}^* \\
\hline
-z_2^* & -z_6^* & -z_{10}^* & z_{35} & z_1^* & 0 & 0 & 0 \\
-z_3^* & -z_7^* & -z_{11}^* & -z_{34} & 0 & z_1^* & 0 & 0 \\
-z_4^* & -z_8^* & -z_{12}^* & z_{33} & 0 & 0 & z_1^* & 0 \\
-z_5^* & -z_9^* & -z_{13}^* & -z_{32} & 0 & 0 & 0 & z_1^* \\
\hline
-z_6 & z_2 & 0 & 0 & 0 & z_{14} & z_{15} & z_{16} \\
-z_{10} & 0 & z_2 & 0 & 0 & z_{17} & z_{18} & z_{19} \\
0 & -z_{14}^* & -z_{17}^* & z_{31} & -z_3^* & z_2^* & 0 & 0 \\
0 & -z_{15}^* & -z_{18}^* & -z_{30} & -z_4^* & 0 & z_2^* & 0 \\
\hline
0 & -z_{16}^* & -z_{19}^* & z_{29} & -z_5^* & 0 & 0 & z_2^* \\
-z_7 & z_3 & 0 & 0 & -z_{14} & 0 & z_{20} & z_{21} \\
-z_{11} & 0 & z_3 & 0 & -z_{17} & 0 & z_{22} & z_{23} \\
0 & -z_{20}^* & -z_{22}^* & z_{28} & 0 & -z_4^* & z_3^* & 0 \\
\hline
0 & -z_{21}^* & -z_{23}^* & -z_{27} & 0 & -z_5^* & 0 & z_3^* \\
-z_8 & z_4 & 0 & 0 & -z_{15} & -z_{20} & 0 & z_{24} \\
-z_{12} & 0 & z_4 & 0 & -z_{18} & -z_{22} & 0 & z_{25} \\
0 & -z_{24}^* & -z_{25}^* & z_{26} & 0 & 0 & -z_5^* & z_4^* \\
\hline
-z_9 & z_5 & 0 & 0 & -z_{16} & -z_{21} & -z_{24} & 0 \\
-z_{13} & 0 & z_5 & 0 & -z_{19} & -z_{23} & -z_{25} & 0 \\
0 & -z_{10} & z_6 & 0 & 0 & z_{26} & z_{27} & z_{28} \\
z_{14}^* & 0 & -z_{26}^* & -z_{25} & -z_7^* & z_6^* & 0 & 0 \\
\hline
z_{15}^* & 0 & -z_{27}^* & z_{23} & -z_8^* & 0 & z_6^* & 0 \\
z_{16}^* & 0 & -z_{28}^* & -z_{22} & -z_9^* & 0 & 0 & z_6^* \\
0 & -z_{11} & z_7 & 0 & -z_{26} & 0 & z_{29} & z_{30} \\
z_{20}^* & 0 & -z_{29}^* & -z_{19} & 0 & -z_8^* & z_7^* & 0 \\
\hline
z_{21}^* & 0 & -z_{30}^* & z_{18} & 0 & -z_9^* & 0 & z_7^* \\
0 & -z_{12} & z_8 & 0 & -z_{27} & -z_{29} & 0 & z_{31} \\
z_{24}^* & 0 & -z_{31}^* & -z_{17} & 0 & 0 & -z_9^* & z_8^* \\
0 & -z_{13} & z_9 & 0 & -z_{28} & -z_{30} & -z_{31} & 0 \\
\hline
z_{17}^* & z_{26}^* & 0 & z_{24} & -z_{11}^* & z_{10}^* & 0 & 0 \\
z_{18}^* & z_{27}^* & 0 & -z_{21} & -z_{12}^* & 0 & z_{10}^* & 0 \\
z_{19}^* & z_{28}^* & 0 & z_{20} & -z_{13}^* & 0 & 0 & z_{10}^* \\
z_{22}^* & z_{29}^* & 0 & z_{16} & 0 & -z_{12}^* & z_{11}^* & 0 \\
\hline
z_{23}^* & z_{30}^* & 0 & -z_{15} & 0 & -z_{13}^* & 0 & z_{11}^* \\
z_{25}^* & z_{31}^* & 0 & z_{14} & 0 & 0 & -z_{13}^* & z_{12}^* \\
z_{26} & -z_{17} & z_{14} & 0 & 0 & 0 & z_{32} & z_{33} \\
0 & 0 & -z_{32}^* & z_{13} & z_{20}^* & -z_{15}^* & z_{14}^* & 0 \\
\hline
0 & 0 & -z_{33}^* & -z_{12} & z_{21}^* & -z_{16}^* & 0 & z_{14}^* \\
z_{27} & -z_{18} & z_{15} & 0 & 0 & -z_{32} & 0 & z_{34} \\
0 & 0 & -z_{34}^* & z_{11} & z_{24}^* & 0 & -z_{16}^* & z_{15}^* \\
z_{28} & -z_{19} & z_{16} & 0 & 0 & -z_{33} & -z_{34} & 0 \\
\hline
0 & z_{32}^* & 0 & -z_9 & z_{22}^* & -z_{18}^* & z_{17}^* & 0 \\
0 & z_{33}^* & 0 & z_8 & z_{23}^* & -z_{19}^* & 0 & z_{17}^* \\
0 & z_{34}^* & 0 & -z_7 & z_{25}^* & 0 & -z_{19}^* & z_{18}^* \\
z_{29} & -z_{22} & z_{20} & 0 & z_{32} & 0 & 0 & z_{35} \\
\hline
0 & 0 & -z_{35}^* & -z_{10} & 0 & z_{24}^* & -z_{21}^* & z_{20}^* \\
z_{30} & -z_{23} & z_{21} & 0 & z_{33} & 0 & -z_{35} & 0 \\
0 & z_{35}^* & 0 & z_6 & 0 & z_{25}^* & -z_{23}^* & z_{22}^* \\
z_{31} & -z_{25} & z_{24} & 0 & z_{34} & z_{35} & 0 & 0 \\
\hline
-z_{32}^* & 0 & 0 & z_5 & z_{29}^* & -z_{27}^* & z_{26}^* & 0 \\
-z_{33}^* & 0 & 0 & -z_4 & z_{30}^* & -z_{28}^* & 0 & z_{26}^* \\
-z_{34}^* & 0 & 0 & z_3 & z_{31}^* & 0 & -z_{28}^* & z_{27}^* \\
-z_{35}^* & 0 & 0 & -z_2 & 0 & z_{31}^* & -z_{30}^* & z_{29}^*
\end{array}\right) \tag{1}
$$

consider the inverse permutations of the columns that are necessary to restore $\boldsymbol{G}$ to its initial arrangement. Restoring $\boldsymbol{G}$ to its initial arrangement may involve any series of column rearrangements involving the swapping of columns between $\boldsymbol{G}_L$ and $\boldsymbol{G}_R$, the rearranging of columns exclusively within $\boldsymbol{G}_L$, and/or the rearranging of columns exclusively within $\boldsymbol{G}_R$.

By Corollary 2.7, any column rearrangement involving swapping columns in $\boldsymbol{G}_L$ with columns in $\boldsymbol{G}_R$ either results in the restoration of $\boldsymbol{B}_1$ form or the elimination of the current $\boldsymbol{B}_1$ form and the creation of a new $\boldsymbol{B}_i$ form with $i \neq 1$. Then, by Lemma 2.5, any rearrangement of columns of $\boldsymbol{G}$ wholly within $\boldsymbol{G}_L$ or wholly within $\boldsymbol{G}_R$ will keep $\boldsymbol{G}$ in the current $\boldsymbol{B}_i$ form. It follows that any series of any types of column rearrangements, in any order, will move $\boldsymbol{G}$ to some $\boldsymbol{B}_i$ form, for some $1 \leq i \leq k$. Hence, given an arbitrary initial arrangement of $\boldsymbol{G}$, the design $\boldsymbol{G}$ is in $\boldsymbol{B}_i$ form for some $1 \leq i \leq k$. □

APPENDIX B

See (1) on the preceding page.

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